Self-Test 1: 4.10-5.2

1. Antidifferentiate. Unless specified, find the most general antiderivative.

   (a) $f(x) = 3\sqrt{x} - \frac{1}{\sqrt{x}}$
   Rewrite, so that $f(x) = 3x^{1/2} - x^{-1/2}$, and

   $$F(x) = 3 \cdot \frac{2}{3}x^{3/2} - \frac{2}{1}x^{1/2} + C = 2x^{3/2} - 2x^{1/2} + C$$

   (b) $f(x) = x^2 + x^{-1}$
   Remember that $\frac{1}{x}$ uses a special antiderivative!

   $$F(x) = \frac{1}{3}x^3 + \ln|x| + C$$

   (and remember the absolute value sign)

   (c) $g'(x) = \frac{4}{\sqrt{1-x^2}}$, $g(1/2) = 1$
   First, the antiderivative is:

   $$g(x) = 4\sin^{-1}(x) + C$$

   and now we'll need to find C. We can compute $\sin^{-1}(1/2)$ by remembering that $1/2$ is from a special triangle- the 30-60-90 triangle. We look for the angle whose opposite side has length 1, and whose hypotenuse is 2- the angle is 30 degrees, or $\frac{\pi}{6}$ radians (remember- all our answers should be in radians).
   Therefore,

   $$1 = 4 \cdot \frac{\pi}{6} + C \Rightarrow C = 1 - \frac{2\pi}{3}$$

   and our final answer is given by:

   $$g(x) = 4\sin^{-1}(x) + \left(1 - \frac{2\pi}{3}\right)$$

2. The following questions refer to $f(x) = x - 1$, $2 \leq x \leq 4$

   (a) Find the exact area under the curve using geometry.

   We can use the area of triangles to do this. Think about this region as the area of the triangle whose base runs from $x = 1$ to

$x = 4$ (this area is $\frac{1}{2} \cdot 3 \cdot 3$) and subtract the triangle whose base runs from $x = 1$ to $x = 2$ (this area is $\frac{1}{2} \cdot 1 \cdot 1$). So overall, the area is

$$\frac{9}{2} - \frac{1}{2} = 4$$

(b) Estimate the area under the curve using 4 rectangles and right endpoints and equally spaced subintervals.

Using 4 rectangles, each width is $\frac{1}{2}$ and the right endpoints are: $2 + \frac{1}{2}, 3, 3 + \frac{1}{2}, 4$. Put these into $f(x) = x - 1$ to find the heights, add the areas up:

$$\frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{5}{2} + \frac{1}{2} \cdot 3 = \frac{9}{2}$$

(c) Write the exact area as a limit, using right endpoints and equally spaced subintervals.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{2i}{n} - 1\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

(d) Write the exact area as a limit, using left endpoints and equally spaced subintervals.

The left endpoints will be:

$$2, 2 + \frac{2}{n}, 2 + 2 \cdot \frac{2}{n}, \ldots, 4$$

So, if we want i to start at 1, then the i^{th} rectangle will have $2 + (i - 1)\frac{2}{n}$ as its left endpoint.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + (i-1)\frac{2}{n} - 1\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + (i-1)\frac{2}{n}\right)\left(\frac{2}{n}\right)$$

(e) Find the area under the curve by computing the limit that you wrote down in part (c).

First, we'll simplify the Riemann sum, then we'll take the limit:

$$\sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) = \frac{2}{n}\left(\sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{2i}{n}\right) =$$

$$\frac{2}{n}\left(n+\frac{2}{n}\sum_{i=1}^{n} i\right) = \frac{2}{n}\left(n+\frac{2}{n}\cdot\frac{n(n+1)}{2}\right) = 2+2\cdot\frac{n(n+1)}{n^2}$$

And finally, taking the limit as $n \to \infty$, we see that:

$$\lim_{n\to\infty}\frac{n(n+1)}{n^2} = 1$$

and our final answer is $2+2=4$, the same as before!

3. Express the following limit as a definite integral:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{2+\frac{5i}{n}}\cdot\frac{5}{n}$$

The width of the interval is 5. As we said in class, there are multiple ways to answer, depending on what you will call your right endpoints. Here are some valid answers:

   (a) If the right endpoints are $2+\frac{5i}{n}$, then the integral is: $\int_2^7 \sqrt{x}\,dx$

   (b) If the right endpoints are $\frac{5i}{n}$, then the integral is: $\int_0^5 \sqrt{2+x}\,dx$

   (c) If the right endpoints are $1+\frac{5i}{n}$, then the integral is: $\int_1^6 \sqrt{1+x}\,dx$

4. Express the following integral as a limit, using right endpoints and equally spaced intervals:

$$\int_1^4 \sin(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\cdot\sin\left(1+\frac{3i}{n}\right)$$

5. Find the value of the sum:

$$\sum_{k=1}^{n}(3k+5) = 3\sum_{k=1}^{n}k+\sum_{k=1}^{n}5 = 3\cdot\frac{n(n+1)}{2}+5n$$