Review Problems: 7.8 and 11.1-11.10

1. What does it mean to say that a series "converges" (I'm looking for the definition; be sure you define any notation you use).

2. If a series converges by the Integral Test, how do you estimate it's sum (or more precisely, how do you estimate the remainder, R_n)?

3. If a series converges by the Alternating Series Test, how do you estimate its sum (or more precisely, how do you estimate the remainder, R_n)?

4. If a series has radius of convergence ρ, can you predict the radius of convergence of the derivative of the series? For the antiderivative?

5. Does the given sequence or series converge or diverge? If the series converges, is it absolute or conditional?

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

   (b) $\left\{\frac{n}{1+\sqrt{n}}\right\}$

   (c) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2+1}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

   (e) $\displaystyle\sum_{n=1}^{\infty} (-6)^{n-1} 5^{1-n}$

   (f) $\left\{\frac{n!}{(n+2)!}\right\}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n n!}$

   (h) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n+2^n}{6^n}$

   (i) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

   (k) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$

   (l) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

   (m) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

6. Evaluate the integral or show it diverges:

   (a) $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx$

   (b) $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$

   (c) $\displaystyle\int_0^{\infty} x^3 e^{-x^4}\,dx$

7. Show that the integral $\displaystyle\int_1^{\infty} \frac{\sin^2(x)}{x^2}\,dx$ converges or diverges. HINT: Do not try to compute the antiderivative. Be clear as to your justification.

8. Find the sum of the series

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{2n}}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{3^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n!)}$

9. Find the radius of convergence. For the last two, include the interval of convergence.

(a) $\sum \frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$ (b) $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$ (c) $\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$

10. Use a series to evaluate the following limit: $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

11. Use a known template series to find a series for the following:

    (a) $\frac{x^2}{1+x}$ (b) $\sin(x^2)$ (c) $x\mathrm{e}^{2x}$

12. Find the Taylor series for $f(x)$ centered at the given base point:

    (a) $x^4 - 3x^2 + 1$, at $x = 1$
    (b) $1/\sqrt{x}$ at $x = 9$ (just get the first four non-zero terms of the Taylor series).
    (c) x^{-2} at $x = 1$. In this case, find a pattern for the n^{th} coefficient so that you can write the series in $\Sigma-$notation. Using this answer, find the radius of convergence.

13. True or False, and give a short reason:

    (a) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ is convergent.
    (b) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
    (c) The Ratio Test can be used to determine if a $p-$series is convergent.
    (d) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
    (e) If $a_n > 0$ for all n and $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges.

14. It is well known that there is no "simple" antiderivative for e^{-x^2}. Find a series representation for $\int \mathrm{e}^{-x^2}\,dx$ and give the radius of convergence. HINT: Start with a template series that we know.

15. Suppose that $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x = 3$ and diverges when $x = -2$.

    (a) What is the largest interval for x on which we can guarantee that the series converges.
    (b) What can be said about the sum: $\sum_{n=0}^{\infty}(-1)^n c_n$
    (c) What can be said about the sum: $\sum_{n=0}^{\infty} c_n 4^n$

16. Find the Maclaurin series for $\ln(x+1)$ and find the radius of convergence. You may do it from scratch or by using a template series.

17. Find the sum: $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

18. Use the remainder for the Taylor series to approximate how large the error will be if I use a 3rd order ($n = 3$) Maclaurin series to estimate $\sin(x)$ at $x = 1/2$.

19. Let $a_n = \frac{2n}{3n+1}$

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

20. Put the following quantities in order, from smallest to largest, if $f(x)$ is a positive, continuous, decreasing function, $a_n = f(n)$, and R_n is the remainder after using n terms of the sum:

$$R_n \qquad \int_n^\infty f(x)\,dx \qquad \int_{n+1}^\infty f(x)\,dx$$

21. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

(a) Show that the series converges absolutely by using the Integral Test (if appropriate- Check it).

(b) Give an estimate of the error using the integral if we use 4 terms to estimate the sum.

22. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

(a) Prove the series converges by using the Alternating Series Test. Be clear about what you have to check for this!

(b) Given that the first few values are given by the following table, how many terms should we use if we want to estimate the sum correct to 3 decimal places?

	$(-1)^n/(n^4)$
$n = 1$	-1
$n = 2$	0.0625
$n = 3$	-0.01234567
$n = 4$	0.00390625
$n = 5$	-0.0016
$n = 6$	0.00077160
$n = 7$	-0.00041649
$n = 8$	0.00024414
$\vdots$	

23. The terms of a series are defined recursively by the equations:

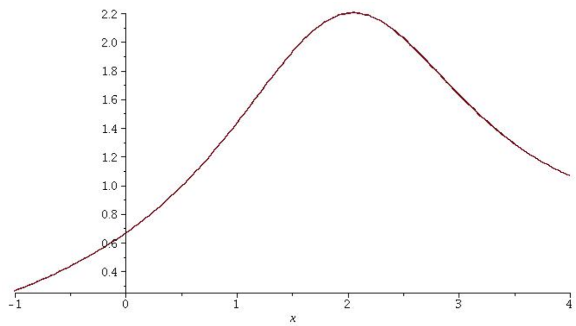
$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

so, for example,

$$a_2 = \frac{6}{7}a_1 = \frac{12}{7}, \qquad a_3 = \frac{11}{11} \cdot a_2 = 1 \cdot \frac{12}{7} = \frac{12}{7}, \cdots$$

Does the series converge or diverge? (Hint: You have enough information to run a convergence test).

24. Consider the graph of the function $f(x)$ below:



(a) Explain why the following is NOT the Taylor series for f centered at $x = 1$:

$$1.4 - (x - 1) + 0.2(x - 1)^2 - 0.2(x - 1)^3 + \cdots$$

(b) Explain why the following is NOT the Taylor series for f centered at $x = 2$:

$$2.2 + 0.1(x - 2) + (x - 2)^2 + 0.5(x - 2)^4 + \cdots$$