

## Selected Solutions, Section 5.2

1. This is good practice in taking *left* endpoints.

In this case,  $f(x) = 3 - x/2$ , and the interval is  $[2, 14]$ . The Riemann sum using 6 rectangles will use:

- Width of each rectangle:  $(14 - 2)/6 = 12/6 = 2$ .
- The height of the rectangles will be evaluated at left endpoints. Subdividing the interval using 6 equal subintervals gives us endpoints:

$$2, 4, 6, 8, 10, 12, 14$$

so that “left endpoints” are 2, 4, 6, 8, 10, 12.

- The area is computed as  $f(x_i^*) = 3 - x_i^*/2$ , or using left endpoints:

$$(2 + 1 + 0 - 1 - 2 - 3) \cdot 2 = -6$$

8. The table is used to estimate  $\int_3^9 f(x) dx$  using 3 equal subintervals. The intervals then have endpoints

$$3, 5, 7, 9$$

- (a) Using right endpoints, the area is approximately:

$$((-0.6) + (0.9) + (1.8))2 = 4.2$$

Since  $f$  is increasing, right endpoints give an overestimate.

- (b) Using left endpoints, the area is approximately:

$$((-3.4) + (-0.6) + (0.9))2 = -6.2$$

Left endpoints give an underestimate for an increasing function.

- (c) Using midpoints:

$$((-2.1) + (0.3) + (1.4))2 = -0.8$$

This is probably more accurate than the other two.

18. For problems 17-19, we want to compare what is given to the formula in the text,

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

In this particular case, the expression looks like

$$\int_{\pi}^{2\pi} \frac{\cos(x)}{x} dx$$

23. Before getting started, note that the widths are  $(0 - -2)/n = 2/n$ , and the “ $i^{\text{th}}$  right endpoint” is  $-2 + 2i/n$ . Therefore,

$$\int_{-2}^{15} (x^2 + x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \left( -2 + \frac{2i}{n} \right)^2 + \left( -2 + \frac{2i}{n} \right) \right) \cdot \frac{2}{n}$$

Simplify those heights first by expanding the product out- we can factor  $2/n$  out of the sum. A little algebra shows that

$$\left[ \left( -2 + \frac{2i}{n} \right)^2 + \left( -2 + \frac{2i}{n} \right) \right] = 2 - \frac{6i}{n} + \frac{4i^2}{n^2}$$

Bringing in the sum:

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left( 2 - \frac{6i}{n} + \frac{4i^2}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{2}{n} \left[ \sum_{i=1}^n 2 - \frac{6}{n} \sum_{i=1}^n i + \frac{4}{n^2} \sum_{i=1}^n i^2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[ 2n - \frac{6}{n} \cdot \frac{n(n+1)}{2} + \frac{4}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} \right] = 4 - 6 + \frac{8}{3} = \frac{2}{3} \end{aligned}$$

27. This exercise is very similar to the last one. Using right endpoints, the width of each rectangle is  $(b-a)/n$ . The  $i^{\text{th}}$  right endpoint is  $a + i(b-a)/n$ , and in this case,  $f(x) = x$ . Therefore,

$$\begin{aligned} \int_a^b x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( a + \frac{(b-a)i}{n} \right) \left( \frac{b-a}{n} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left[ \sum_{i=1}^n a + \frac{b-a}{n} \sum_{i=1}^n i \right] = \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left[ an + \frac{b-a}{n} \frac{n(n+1)}{2} \right] = \\ &= \lim_{n \rightarrow \infty} \left[ a(b-a) + \frac{(b-a)^2}{2} \cdot \frac{n+1}{n} \right] = \\ &= a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2 - a^2}{2} \end{aligned}$$

37. For #37, recall that  $x^2 + y^2 = r^2$  is a circle of radius  $r$  centered at the origin. Solving for  $y$ , we could express the upper half of a circle of radius  $r$  as:

$$y = \sqrt{r^2 - x^2}$$

Therefore, the function  $f(x) = 1 + \sqrt{9 - x^2}$  is the upper half of a circle of radius 3 (centered at the origin) and shifted up one unit. The integral is using the interval  $[-3, 0]$ , so the area in this case,

$$\int_{-3}^0 (1 + \sqrt{9 - x^2}) dx$$

is the area of 1/4 of the circle, plus the rectangle of height 1:

$$\int_{-3}^0 (1 + \sqrt{9 - x^2}) dx = 1 \cdot 3 + \frac{1}{4} \pi 3^2 = 3 + \frac{9}{4} \pi$$

53. We want to re-write the integral as

$$\int_{-4}^3 f(x) dx + 2 \int_{-4}^3 x dx + \int_{-4}^3 5 dx$$

and use geometry for the last two integrals. The first/last are easy to compute. For the middle one, draw a sketch and use the area of a triangle. You should find that (in order) the values are:

$$-3 - 12 + 30 = 15$$

55. For this exercise, note that  $f(x) = x^2 - 4x + 4 = (x - 4)^2$ , so that  $f(x) \geq 0$ . Therefore,

$$\int_0^4 f(x) dx \geq 0 \cdot (4 - 0) = 0$$

59. To use Property 8, we have to find the minimum  $m$  and maximum  $M$  of  $\sqrt{x}$  on the interval  $[1, 4]$ . Since the function is increasing, the minimum is  $\sqrt{1} = 1 = m$  and the maximum is  $\sqrt{4} = 2 = M$ . Therefore,

$$1(4 - 1) \leq \int_1^4 \sqrt{x} dx \leq 2(4 - 1)$$

or

$$3 \leq \int_1^4 \sqrt{x} dx \leq 6$$