Antiderivatives

1. The process of differentiation converts a function to another function: $f \to f'$

   Antidifferentiation is the backwards process: $f \to F$, so that $F' = f$.

2. While the derivative was defined in terms of a limit:

   $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

   the antiderivative is defined in terms of its solution. In other words, computing an antiderivative can be much different. In fact, some antiderivatives cannot be written as a simple rule (like a polynomial or trig function). Initially, we will focus on the "nice" antiderivatives, found in Table 2.

3. Rules for computing antiderivatives:

   - $cf(x) \to cF(x)$, where $F' = f$. Example: $3x^{\frac{1}{2}} \to 3 \cdot \frac{2}{3}x^{\frac{3}{2}} + C$
   - $f(x) + g(x) \to F(x) + G(x)$. Example: $x^2 + x^{\frac{1}{2}} \to \frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + C$.

4. General versus specific antiderivatives. Imagine drawing a function f from its derivative function, f'. While f' gives us the local slope information (so f is increasing or decreasing), it does not give us *position* information. Therefore, if you have one antiderivative, f, any vertical shift of the graph gives another antiderivative.

   The general antiderivative of f is a *family* of functions, $F + c$, where $F' = f$.

   If we specify that $F(x_0) = y_0$, then antidifferentiation produces *ONE* function F.

   Example: $f = x^2 + \cos(x)$, $F(0) = 1$. Then the general antiderivative is

   $$F(x) = \frac{1}{3}x^2 + \sin(x) + C$$

   which is a family of curves. The value of C is found by substituting the requirement $F(0) = 1$ into the expression:

   $$1 = \frac{1}{3}0^2 + \sin(0) + C \Rightarrow C = 1$$

   So the specific antiderivative is $F(x) = \frac{1}{3}x^3 + \sin(x) + 1$

5. Worked Example: Find the antiderivative:

   $$f''(x) = 20x^3 - 10, f(1) = 1, f'(1) = -5$$

   By Table 2,

   $$f'(x) = 20 \cdot \frac{1}{4}x^4 - 10x + C = 5x^4 - 10x + C$$

   To find the constant, use $f'(1) = -5$:

   $$-5 = 5(1)^4 - 10(1) + C \Rightarrow -10 = -10 + C \Rightarrow C = 1$$

   so that

   $$f'(x) = 5x^4 - 10x + 1$$

   Now,

   $$f(x) = x^5 - 5x^2 + x + C$$

   with $f(1) = 1$, so:

   $$1 = 1 - 5 + 1 + C$$

   so

   $$f(x) = x^5 - 5x^2 + x + 4$$

6. Worked Example: Find the general antiderivative:

$$g(t) = \frac{t^3 + 2t^2}{\sqrt{t}}$$

We don't have a formula to deal with a fraction. We need to get rid of the fraction by writing:

$$\frac{t^3 + 2t^2}{\sqrt{t}} = \frac{t^3}{\sqrt{t}} + \frac{2t^2}{\sqrt{t}} = \frac{t^3}{t^{1/2}} + \frac{2t^2}{t^{1/2}} = t^{3-1/2} + 2t^{2-1/2} = t^{5/2} + 2t^{3/2}$$

Now we can use the Table. The general antiderivative is:

$$G(t) = \frac{2}{7}t^{7/2} + \frac{4}{5}t^{5/2} + C$$

7. Worked Example: Find the general antiderivative:

$$f(\theta) = \mathrm{e}^{\theta} + \sec(\theta)\tan(\theta)$$

Solution:

$$F(\theta) = \mathrm{e}^{\theta} + \sec(\theta) + C$$

8. Worked Example: Given that the graph of f passes through the point $(1, 6)$ and the slope of its tangent line at $(x, f(x))$ is $2x + 1$, find $f(2)$.

Solution:

First, interpret the problem as being: Given $f'(x) = 2x+1$ and $f(1) = 6$, find $f(2)$. Since $f'(x) = 2x+1$, $f(x) = x^2 + x + C$. Solve for C: $6 = 1^2 + 1 + C$ so that $C = 4$. Now $f(x) = x^2 + x + 4$, so $f(2) = 2^2 + 2 + 4 = 10$.

9. GENERAL NOTES:

The table does not tell us how to deal with products or fractions of functions (like Example 6 above). Therefore, we have to do some algebraic manipulation first.

If we're not given an initial condition (i.e., $f(1) = 2$), then the general antiderivative *always* has an arbitrary constant.

10. Make some flashcards for the following (which is Table 2 in your book). We'll want to begin memorizing them!

Denote the antiderivative of given functions f, g by F, G respectively.

function	Antiderivative	Description
$cf(x)$	$cF(x)$	Constants factor out
$f(x) + g(x)$	$F(x) + G(x)$	Sum Rule
$x^n, n \neq 1$	$\frac{1}{n+1}x^{n+1}$	Power Rule
$\frac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\cos(x)$	$\sin(x)$	
$\sin(x)$	$-\cos(x)$	
$\sec^2(x)$	$\tan(x)$	
$\sec(x)\tan(x)$	$\sec(x)$	
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$	
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$	