Final Exam Pack A

1. Short Answer/True or False. You may assume that all vectors are in $\mathbb{R}^3$.

   (a) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.
   FALSE, however it would have been true if $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

   (b) $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}'(t)$
   FALSE: To differentiate, use a rule like the product rule: $\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$

   (c) There is a vector field whose curl is given by $\langle x, y, z\rangle$.
   FALSE. To check, we know that $\text{div}(\text{curl}(\mathbf{F})) = 0$, but in our case, the divergence is 3.

2. A constant force $\mathbf{F} = \langle 3, 5, 10\rangle$ moves an object along the line segment from $(1, 0, 2)$ to $(5, 3, 8)$. Find the work done if the distance is in meters and the force is measured in newtons.

   SOLUTION: $\langle 3, 5, 10\rangle \cdot \langle 5 - 1, 3 - 0, 8 - 2\rangle = 97$

3. If $u = \sqrt{r^2 + s^2}$, $r = y + x\cos(t)$ and $s = x + y\cos(t)$, compute u_x and u_t at $x = 1, y = 2$ and $t = 0$.
   SOLUTION:

$$u_x = u_r r_x + u_s s_x \qquad \text{Evaluating, we get: } \sqrt{2}$$
$$u_t = u_r r_t + u_s s_t \qquad \text{Evaluating, we get: } 0$$

4. Let vector $\mathbf{a} = \langle 1, 1, -2\rangle$ and $\mathbf{b} = \langle 3, 2, -1\rangle$.

   (a) Find the area of the parallelogram formed using $\mathbf{a}, \mathbf{b}$ (as position vectors).
   SOLUTION: The area is $|\mathbf{a} \times \mathbf{b}| = |\langle 3, -5, -1\rangle| = \sqrt{35}$

   (b) Find $\text{Proj}_{\mathbf{b}}(\mathbf{a})$
   SOLUTION:

$$\text{Proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b} = \frac{1}{2}\langle 3, 2, -1\rangle$$

5. Find the set of points for which $g(x, y) = \ln(x^2 + y^2 - 4)$ is continuous.

   SOLUTION: $x^2 + y^2 > 4$

6. Find the equation of a plane (normal form, not parametric form), if the plane goes through points $(3, -1, 1)$, $(4, 0, 2)$ and $(6, 3, 1)$.

   SOLUTION: We wanted you to compute the normal for the plane, which is found by taking two vectors in the plane and computing the cross product. For example, we might take

$$\mathbf{v}_1 = \langle 4 - 3, 0 - (-1), 2 - 1\rangle \text{ and } \mathbf{v}_2 = \langle 6 - 4, 3 - 0, 1 - 2\rangle$$

   With these two vectors, the cross product is $\langle -4, 3, 1\rangle$. Now take any of the points and form the associated equation for the plane. For example, taking the first point, we have:

$$-4(x - 3) + 3(y + 1) + (z - 1) = 0$$

7. Suppose an object starts at the origin with initial velocity $\langle 1, -1, 3\rangle$, and its acceleration is given by $\mathbf{a}(t) = \langle 6t, 12t^2, -6t\rangle$. Find its position function.

   SOLUTION: Integrate twice. We are given information from which we can determine the constants of integration.

$$\mathbf{v}(t) = \langle 3t^2 + C_1, 4t^3 + C_2, -3t^2 + C_3\rangle$$

   Substituting $t = 0$ and equating this with our initial velocity, we get $C_1 = 1, C_2 = -1$ and $C_3 = 3$.

$$\mathbf{v}(t) = \langle 3t^2 + 1, 4t^3 - 1, -3t^2 + 3\rangle$$

From which we get our position function

$$\mathbf{r}(t) = \langle t^3 + t + C_4, t^4 - t + C_5, -t^3 + 3t + C_6 \rangle$$

Since we start at the origin, all constants are zero:

$$\mathbf{r}(t) = \langle t^3 + t, t^4 - t, -t^3 + 3t \rangle$$

8. Let $f(x, y) = \sqrt[3]{x^3 + y^3}$. Is f differentiable at the origin? Hint: Compute $f_x(0, 0)$ using the *definition* and compare to the derivative using the regular rules of differentiation.

   SOLUTION: Using the rules of differentiation, we get that

$$f_x(x, y) = \frac{x^2}{(x^3 + y^3)^{2/3}}$$

   Using the definition,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h^3} - 0}{h} = 1$$

   (Recall that $\sqrt[3]{h^3} = h$ for all real numbers h). We note that f_x is not continuous at the origin, so f is not differentiable there.

9. Over a certain region, the temperature at a point (x, y, z) is given by $T(x, y, z) = 5x^2 - 3xy + xyz$.

   (a) Find the rate of change of the temperature at $(1, 2, 1)$ in the direction of $\langle 1, 1, 1 \rangle$.
   SOLUTION: This is the directional derivative.

$$D_u T = \nabla T \cdot \mathbf{u} = \langle 6, -2, 2 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

   (Its fine if you left it as $6/\sqrt{3}$).

   (b) In which direction does T increase most?
   SOLUTION: The direction of greatest increase is in the direction of the gradient, $\nabla T(1, 2, 1)$, which is

$$\mathbf{u} = \frac{1}{\sqrt{44}} \langle 6, -2, 2 \rangle$$

   (c) What is the maximum rate of change of T?
   SOLUTION: The maximum rate of change of T is in the direction of ∇T. Moving in that direction, $D_u f = |\nabla T| = \sqrt{44}$

10. Find the absolute maximum and minimum values of f on the set D: $f(x, y) = x + y - xy$, where D is the closed triangular region with vertices at $(0, 0), (0, 2)$ and $(4, 0)$.

    SOLUTION: For the absolute (or global) max and min, we check f at the critical points and boundary points. For the critical points,

$$f_x = 1 - y = 0 \quad \Rightarrow \quad y = 1 \qquad \text{and} \qquad f_y = 1 - x = 0 \quad \Rightarrow \quad x = 1$$

    Since they both must be zero at the same point, that point is $(1, 1)$, and the value of $f(1, 1) = 1$.

    We note that the boundary is a right triangle, so we evaluate the three legs separately:

    - $x = 0, 0 \leq y \leq 2$ On this edge, $f(y) = y$. No critical points on this edge, so just evaluate at the endpoints: $f(0) = 0$ and $f(2) = 2$.
    - $y = 0,\ 0 \leq x \leq 4$. On this edge, $f(x) = x$. No critical points, so evaluate at the endpoints: $f(0) = 0,\ f(4) = 4$.

- On the hypotenuse, $y = 2 - \frac{1}{2}x$. Substituting, we get $f(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 2$, with $0 \le x \le 4$. The critical point is $x = 3/2$, and $f(3/2) = 7/8$. We also get $f(0) = 2$ and $f(4) = 4$.

For the max, we get a value of 4 at the point $(4, 0)$. For the minimum, we get a value of 0 at $(0, 0)$.

11. Evaluate by first reversing the order of integration: $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx$

    (TYPO: The original was missing the dx term)

    SOLUTION: Sketch the region, which is above $y = \sqrt{x}$ and below $y = 2$, so that we get:

$$\int_0^2 \int_0^{y^2} \frac{1}{1+y^3}\, dx\, dy = \int_0^2 \frac{y^2}{1+y^3}\, dy = \left(\frac{1}{3}\ln(1+y^3)\right)\Big|_0^2 = \frac{1}{3}\ln(9) = \frac{2}{3}\ln(3)$$

    (The last equality wasn't necessary, but do you see how it was done?)

12. Convert the following integral to polar coordinates (do not evaluate): $\int_0^1 \int_y^{\sqrt{2-y^2}} x + y\, dx\, dy$

    SOLUTION: Draw a sketch of the region. You should find that the region is the pie-shaped wedge, where $0 \le r \le \sqrt{2}$ and $0 \le \theta \le \pi/4$. These then go into the bounds for the integral:

$$\int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos(\theta) + r\sin(\theta)) r\, dr\, d\theta$$

13. Given the solid E that is bounded by the cylinder $x^2 + y^2 = 4$, and the planes $z = 0$ and $y + z = 3$:

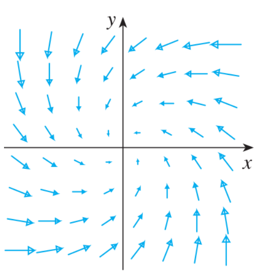
    (a) Find a parametric representation for the curve of intersection between the cylinder and $y + z = 3$:
    SOLUTION: Since x, y are on a circle, it's easiest to parametrize them as: $x = 2\cos(t), y = 2\sin(t)$. Then $z = 3 - y = 3 - 2\sin(t)$. Therefore,

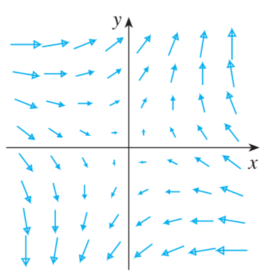
$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 3 - 2\sin(t)\rangle$$

    (b) Find a triple integral (and evaluate it) for the volume of E.
    SOLUTION: Probably easiest to use cylindrical coordinates. Therefore, we'll integrate over the region where $0 \le \theta \le 2\pi$ and $0 \le r \le 2$, then the height will go from $z = 0$ to $z = 3 - r\sin(\theta)$

$$\int_0^{2\pi} \int_0^2 \int_0^{3 - r\sin(\theta)} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 3r - r^2\sin(\theta)\, dr\, d\theta = \int_0^{2\pi} 6 - \frac{8}{3}\sin(\theta)\, d\theta = 12\pi$$

14. For each vector field below, estimate as to whether or not it represents a conservative vector field.





    SOLUTION: We check by drawing a simple closed curve in the vector field, might as well be a unit circle. For the vector field to the left, we see that the line integral (going CCW) will be positive, because the unit tangent on the curve will generally be in the same direction as the vector field. In the vector field to the right, it looks like it could very well be conservative.

15. Set up, but do not evaluate, the surface integral $\iint_S y\,dS$, where S is the surface that consists of $\mathbf{r}(u,v) = \langle uv, u^2, u-2v\rangle$, where $0 \le u \le 1$ and $0 \le v \le 1$.

SOLUTION: For the general surface integral, we need to compute the normal vector to the surface. Putting in $\mathbf{r}_u, \mathbf{r}_v$, respectively gives:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ v & 2u & 1 \\ u & 0 & -2 \end{vmatrix} = \langle -4u, 2v+u, -2u^2\rangle$$

Now we integrate:

$$\int_0^1 \int_0^1 f(u,v) \cdot |\mathbf{r}_u \times \mathbf{r}_v\, dA = \int_0^1 \int_0^1 u^2 \sqrt{16u^2 + (2v+u)^2 + 4u^4}\, du\, dv$$

16. Verify Stokes' Theorem, if $\mathbf{F} = \langle -y, x, -2\rangle$, and the surface S is the cone $z^2 = x^2 + y^2$, $0 \le z \le 4$ oriented downward.

SOLUTION: To verify Stokes' Theorem, we'll need to compute both sides: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$

- The line integral: The boundary of the surface is the circle $x^2 + y^2 = 4^2$, with $z = 4$. Since the normal is pointed downward, we'll traverse the circle in a clockwise fashion. Therefore, the parameterization for the curve will be:

  $$\mathbf{r}(t) = \langle 4\cos(-t), 4\sin(-t), 4\rangle = \langle 4\cos(t), -4\sin(t), 4\rangle$$

  The integrand will be $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$, which is:

  $$\langle 4\sin(t), 4\cos(t), -2\rangle \cdot \langle -4\sin(t), -4\cos(t), 0\rangle = -16\sin^2(t) - 16\cos^2(t) = -16$$

  Therefore the line integral is $(-16)(2\pi) = -32\pi$.

- For the surface integral, our surface is the cone, and we're integrating over the circle of radius 4 centered at the origin in the xy-plane. For the moment, we'll keep our surface normal as $\langle g_x, g_y, -1\rangle$. Note that with $\mathbf{G} = \langle P, Q, R\rangle$, then we're using the surface integral formula (Equation 10, Section 16.7, pg. 1118):

  $$\iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\, dA$$

  In our case, $\mathbf{G}$ is the curl of F:

  $$\text{curl}(\mathbf{F}) = \langle 0, 0, 2\rangle$$

  Therefore, our integrand is -2, and the surface integral reduces to -2 times the area of the circle: $-2(16\pi) = -32\pi$

17. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = \langle xye^z, xy^2z, -ye^z\rangle$ and S is the surface of the box bounded by the coordinate planes and $x = 3, y = 2, z = 1$.

SOLUTION: (Assume orientation is pointing outward)

Using the Divergence Theorem, this is simple to integrate. We first compute the divergence of the vector field, and we find that it is $2xyz$. Then:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 2\int_0^3 x\,dx \int_0^2 y\,dy \int_0^1 z\,dz = 9$$