Final Exam Pack B

1. Short Answer/True or False.

   (a) True/False: The linear equation $Ax + By + Cz + D = 0$ represents a line in space.
   FALSE: This is a plane.

   (b) True/False: If $|\mathbf{r}(t)|^2 = k$ for all t, then $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.
   TRUE: We can differentiate $\mathbf{r}(t) \cdot \mathbf{r}(t)$ to show that (the derivative is zero).

   (c) If $\mathbf{r}(t) = \langle \sqrt{2-t}, \ln(t-1) \rangle$, then find the domain of $\mathbf{r}$, and compute $\mathbf{r}'$.
   SOLUTION: $1 < t \leq 2$ for the domain, and $\mathbf{r}'(t) = \langle \frac{1}{2}(2-t)^{-1/2}(-1), 1/t - 1 \rangle$

   (d) Find the differential of $u = \sqrt{x^2 + 3y^2}$.
   SOLUTION: $du = u_x\, dx + u_y\, dy = \frac{x}{\sqrt{x^2+3y^2}}\, dx + \frac{3y}{\sqrt{x^2+3y^2}}\, dy$

   (e) True/False: If $\int_C \mathbf{F} \cdot d\mathbf{r}$ when C is the unit circle, then $\mathbf{F}$ is conservative.
   FALSE. The integral must be zero for any arbitrary closed curve (not just a circle) before we can conclude that the vector field is conservative.

2. Let vector $\mathbf{a} = \langle 2, 1, 3 \rangle$ and $\mathbf{b} = \langle -1, 2, 0 \rangle$.

   (a) Find the area of the parallelogram formed using $\mathbf{a}, \mathbf{b}$ (as position vectors).
   SOLUTION: The area is given by $|\mathbf{a} \times \mathbf{b}| = |\langle -6, -3, 5 \rangle| = \sqrt{70}$

   (b) Find the component of the projection of $\mathbf{a}$ onto $\mathbf{b}$ in the direction of $\mathbf{b}$. That is, $\text{comp}_{\mathbf{b}}(\mathbf{a})$.
   SOLUTION: $\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = 0$.

3. Find the distance between the plane $2x - 3y + z = 4$ and $4x - 6y + 2z = 3$.

   SOLUTION: First check that the planes are parallel (they are, the normals are parallel and they are not the same plane). Next, get a point on one plane and find the distance from it to the other plane. In this case, $(0, 0, 4)$ is an easy point to compute on the first plane. The distance to the other plane is then:

   $$\frac{|4(0) - 6(0) + 2(4) - 3|}{\sqrt{16 + 36 + 4}} = \frac{5}{\sqrt{56}}$$

   For an extra check, we can choose another point to see if we get the same number.

4. Find the limit, if it exists: $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

   SOLUTION: If we go to zero along either axis or $y = x$, we get zero. However, if we choose $x = y^2$, then we get:

   $$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

   Therefore, the limit overall does not exist at the origin.

5. Reparameterize the curve with respect to arc length: $\mathbf{r}(t) = \langle 2t, 1 - 3t, 5 + 4t \rangle$.

   SOLUTION: We compute s as a function of t, $s = f(t)$, then we invert that $t = f^{-1}(s)$, and we substitute that back in the parameterization.

   $$s = \int_0^t \sqrt{2^2 + (-3)^2 + 4^2}\, dt = \sqrt{29}\, t \quad \Rightarrow \quad t = \frac{s}{\sqrt{29}}$$

   Therefore,

   $$\mathbf{r}(s) = \left\langle 2\frac{s}{\sqrt{29}}, 1 - 3\frac{s}{\sqrt{29}}, 5 + 4\frac{s}{\sqrt{29}} \right\rangle$$

6. Let $f(x,y) = 4 - x^2 - y^2$. Find $f_y(2,1)$ using the *definition* of the partial derivative.

$$f_y(2,1) = \lim_{h\to 0} \frac{f(2,1+h) - f(2,1)}{h} = \lim_{h\to 0} \frac{-(1+h)^2 + 1}{h} = \lim_{h\to 0} -h - 2 = -2$$

7. $u = x\mathrm{e}^{ty}$ where $x = \alpha^2\beta, y = \beta^2\gamma, t = \gamma^2\alpha$. Find $\partial u/\partial\beta$ if $\alpha = -1, \beta = 2, \gamma = 1$.

   SOLUTION: You might draw a tree that gives the relationships between the variables. From that, we see that

$$u_\beta = u_x x_\beta + u_y y_\beta + u_t t_\beta$$

   We want to evaluate each of these at the given values:

$$u_\beta = (\mathrm{e}^{-4})(1) + (-2\mathrm{e}^{-4})(4) + 0 = -7\mathrm{e}^{-4}$$

8. Find the maximum rate of change of f at the given point and the direction in which it occurs: $f(x,y,z) = (x+y)/z$, at $(1,1,-1)$.

   SOLUTION: The maximum rate of change of f will be $|\nabla f|$, and it occurs when we move in the direction of the gradient. In this case,

$$\nabla f = \left\langle \frac{1}{z}, \frac{1}{z}, \frac{-(x+y)}{z^2} \right\rangle \quad \Rightarrow \quad \nabla f(1,1,-1) = \langle -1, -1, -2\rangle$$

   This is the direction to take to achieve the maximum rate of change of f, which is $\sqrt{6}$.

9. Let $f(x,y) = xy$. Find $\nabla f(3,2)$ and use it to find the tangent line to the level curve $f(x,y) = 6$ at the point $(3,2)$. Sketch the level curve, the tangent line and the gradient vector.

   SOLUTION: Sketch the level curve $xy = 6$, and locate $(3,2)$. The gradient vector is orthogonal to the level curve. The gradient is:

$$\nabla f = \langle y, x\rangle|_{(x,y)=(3,2)} = \langle 2, 3\rangle$$

   We can get the slope of the tangent line using the partial derivatives in the gradient:

$$f(x,y) = k \quad \Rightarrow \quad f_x + f_y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{2}{3}$$

   As a check, we see that $y = 6/x$, so $y' = -6/x^2$, and at $x = 3$, the slope is $-2/3$. The equation of the tangent line is then:

$$y - 2 = -\frac{2}{3}(x - 3)$$

10. Find three positive numbers whos sum is 12 and whose product is a maximum.

    SOLUTION 1: Maximize xyz so that $x > 0, y > 0, z > 0$ and $x + y + z = 12$ (we can get rid of zeros, since that is clearly not a max). We use the constraint in order to get a function of two variables:

$$f(x,y) = xy(12 - x - y) = 12xy - x^2y - xy^2$$

    Now we find the critical points:

$$f_x = 12y - 2xy - y^2 = y(12 - 2x - y) = 0 \quad \Rightarrow \quad y = 12 - 2x$$

$$f_y = 12x - x^2 - 2xy = x(12 - x - 2y) = 0 \quad \Rightarrow \quad x = 12 - 2y$$

    From these two equations, we get $x = 4, y = 4, z = 4$. Note again that we ignore zeros, since those will not yield a max.

SOLUTION 2: Using Lagrange Multipliers, we want to maximize $f(x, y, z) = xyz$ such that $x+y+z = 12$, and we'll ignore zeros and negative numbers. Then we get:

$$\begin{array}{rl} \nabla f & = \lambda \nabla g \\ g(x,y,z) & = k \end{array} \quad \Rightarrow \quad \begin{array}{rl} yz & = \lambda \\ xz & = \lambda \\ xy & = \lambda \\ x+y+z & = 12 \end{array}$$

We can throw out $\lambda = 0$. Then the first two equations will imply that $x = y$, and the last two equations imply that $y = z$. Substitute these into g so that

$$3x = 12 \quad \Rightarrow \quad x = 4, \quad y = 4, \quad z = 4$$

11. Sketch the region of integration and change the order: $\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} f(x, y) dx\, dy$

SOLUTION: The region is the right half of a circle of radius 2. We would normally translate this to polar coordinates, but we were asked to switch the order of integration. In that case,

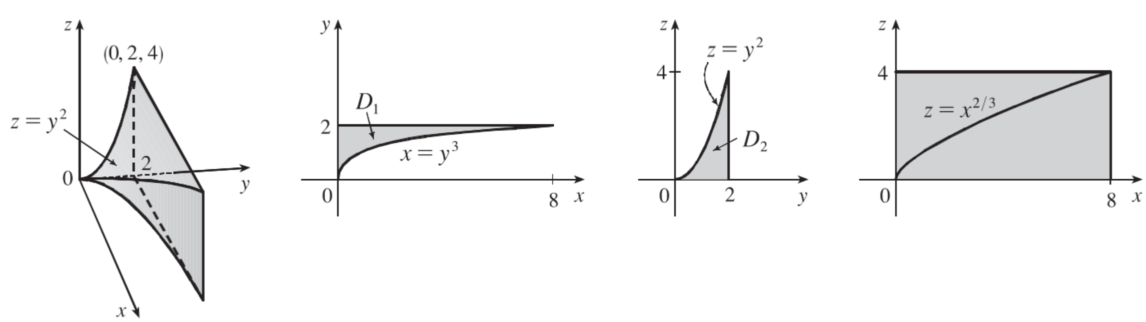
$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\, dy\, dx$$

12. Given the following triple integral, rewrite using two other orders of integration (your choice):

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z)\, dz\, dx\, dy$$

SOLUTION: Draw a sketch of the functions shown, first focus on the xy plane and then z coming up from there. In that case, it is clear how to project the object to the xy plane and the yz plane. What is probably not obvious is the projection to the xz plane.

Find the parameterization of the curve that is the intersection of $x = y^3$ and $z = y^2$, which is $\langle y^3, y, y^2 \rangle$. Projecting this to the xz plane, we just get $\langle y^3, 0, y^2 \rangle$, or writing it in terms of x, z, we get $z = x^{2/3}$. That splits the rectangle into two regions, as shown below.



What are some of the easier ways to rewrite the integral? Leave z first, and switch the order of x, y. The second easiest is probably to take x over the yz plane. We'll include them all below for your enjoyment:

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \int_0^{y^2} f(x, y, z)\, dz\, dy\, dx \qquad \int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} f(x, y, z)\, dx\, dy\, dz \qquad \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x, y, z)\, dx\, dz\, dy$$

And the last two are split in the xz plane:

$$\int_0^8 \int_0^{x^{2/3}} \int_{x^{1/3}}^2 f(x,y,z)\, dy\, dz\, dx + \int_0^8 \int_{x^{2/3}}^4 \int_{\sqrt{z}}^2 f(x,y,z)\, dy\, dz\, dx$$

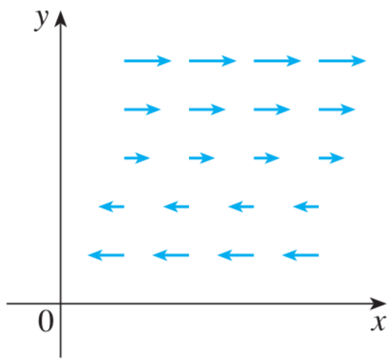
$$\int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{z}}^2 f(x,y,z)\, dy\, dx\, dz + \int_0^4 \int_{x^{3/2}}^8 \int_{x^{1/3}}^2 f(x,y,z)\, dy\, dx\, dz$$

13. Find the work done by the vector field $\mathbf{F} = \langle (1+xy)e^{xy}, x^2 e^{xy} \rangle$ in moving a particle CCW along the ellipse $x^2 + y^2/4 = 1$ (that is, from point $(1,0)$ to $(0,2)$).

    SOLUTION: We find that $\mathbf{F}$ is conservative, and the potential function is $f(x,y) = xe^{xy}$. Using the Fundamental Theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,2) - f(1,0) = 0 - 1 = -1$$

14. For the vector field $\mathbf{F} = \langle P, Q \rangle$ below, estimate the signs (positive, negative, zero) of each: (i) P_x, P_y, (ii) Q_x, Q_y, then (iii) the curl and divergence.



(a) The arrows don't change as x increases, so $P_x = 0$. On the other hand, the arrows start negative and go positive as y increases, so $P_y > 0$.

(b) $Q_x = Q_y = 0$ since there is no Q component.

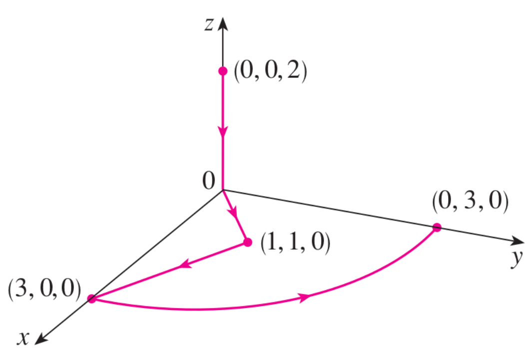
(c) Curl: $Q_x - P_y < 0$. Negative curl, CW rotation.

(d) Divergence: $P_x + Q_y = 0 + 0 = 0$.

15. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, if

$$\mathbf{F} = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z \rangle$$

    And C is the path shown below, starting at $(0,0,2)$ and ending at $(0,3,0)$.



We find that the curl of $\mathbf{F}$ is zero, so $\mathbf{F}$ is conservative. The potential function is

$$f(x,y,z) = x^3yz - 3xy + z^2$$

And we take $f(0,3,0) - f(0,0,2) = -4$.

16. Set up, but do not evaluate, the flux integral for $\mathbf{F} = \langle -y, 0, 1 \rangle$ across the surface $\mathbf{r}(u,v) = \langle uv, u^2, u - 2v \rangle$ where $0 \le u \le 1$ and $0 \le v \le 1$.

    SOLUTION: For the basic flux integral, we'll need the surface normal. We should find that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -4u, u + 2v, -2u^2 \rangle$$

The integrand, $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$, is computed and placed in the integral below:

$$\int_0^1 \int_0^1 4u^3 - 2u^2 \, du \, dv$$

17. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F} = \langle yz, 2xz, \mathrm{e}^{xy} \rangle$, where C is the circle (CCW) $x^2 + y^2 = 16$, $z = 5$.

    SOLUTION: Using Stokes' Theorem, we see that $\text{curl}(\mathbf{F}) = \langle x\mathrm{e}^{xy} - 2x, y - y\mathrm{e}^{xy}, z \rangle$. We also note that the unit normal is especially simple, $\langle -g_x, -g_y, 1 \rangle = \langle 0, 0, 1 \rangle$. Therefore,

$$\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_D \text{curl}(\mathbf{F}) \cdot \langle 0, 0, 1 \rangle dA = \iint_D z \, dA = 5 \iint_D \, dA$$

    The integral represents the area of the disk:

$$5 \times \pi(4)^2 = 80\pi$$

18. Find the work of the vector field $\mathbf{F} = \langle 2z, 8x - 3y, 3x + y \rangle$, moving a particle along C is the boundary of the part of the plane going through (in order) $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 2)$, then back to $(1, 0, 0)$.

    SOLUTION: Setting things up for Stokes' Theorem, we see that the curl is $\langle 1, -1, 8 \rangle$. The surface is the plane. If we look at the equations, we see that the equation of the plane can be written as $z = 2 - 2y - 2x$, so the normal vector, using the upward pointing normal, is:

$$\langle -g_x, -g_y, 1 \rangle = \langle 2, 2, 1 \rangle$$

    Dot the curl with the normal to get a constant value, 8, and integrate over the triangle in the xy plane (whose area is 1/2):

$$8 \iint_{\text{triangle}} dA = 4$$

19. Find the flux of $\mathbf{F} = \langle 3x - 2y + 4z \rangle$ across the surface of the sphere $x^2 + y^2 + z^2 = 9$.

    TYPO: You probably suspected that $\mathbf{F}$ ought to be written as $\langle 3x, -2y, 4z \rangle$, and you would be right.

    This problem is kind of a template divergence theorem problem. Here, we see that the divergence of $\mathbf{F}$ is 5, so that by the Divergence Theorem, the flux across the surface is given by:

$$5 \iint_E dV = 5 \times \text{ Vol of sphere} = 5 \cdot \frac{4}{3}\pi(3)^3 = 180\pi$$