Section 14.6

- Define the Directional Derivative, $D_u f$

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- Tangent Planes/Normal Lines

The Directional Derivative

To measure the (instantaneous) rate of change of $z = f(x, y)$ based at (a, b) in the direction of *unit* vector $\mathbf{u} = \langle u_1, u_2 \rangle$:

$$D_{\mathbf{u}} f(a, b) \doteq \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

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$$D_{\mathbf{u}} f(1, 0) = (2)\left(\frac{1}{\sqrt{2}}\right) + (0)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

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The **gradient of** $f(x,y)$ is denoted by ∇f, and is the vector

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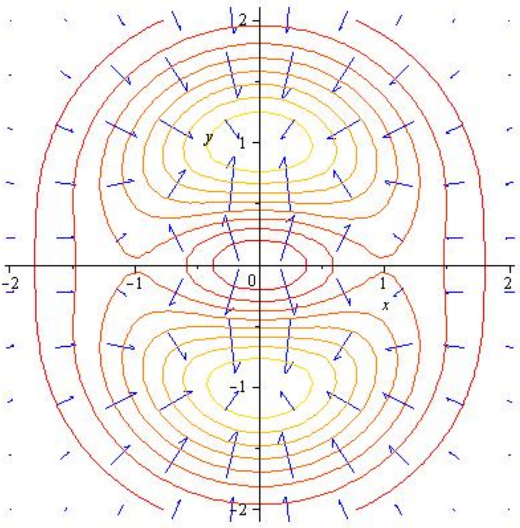
or equivalently

$$\nabla f = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$$

Example: $f(x,y) = (x^2 + 3y)\mathrm{e}^{-x^2-y^2}$



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$$\nabla f(5,2) = \langle f_m(5,2), f_n(5,2) \rangle = \langle 54, 46 \rangle$$

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In this case,

$$\vec{u} = \frac{1}{\sqrt{f_x^2(a,b) + f_y^2(a,b)}}\langle f_x(a,b), f_y(a,b)\rangle$$

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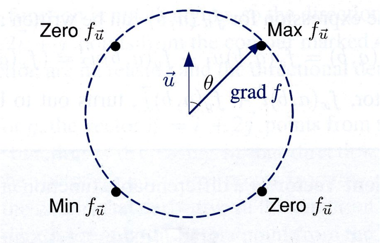
  This is along the level curve (so f does not change, and $D_u f$ should be zero).

Summary of the directions:
Imagine that ∇f is fixed and $\vec{u}$ can rotate. Then:



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   At the origin, the temperature is 100 degrees.

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Therefore, the direction is:

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This points back towards the origin!

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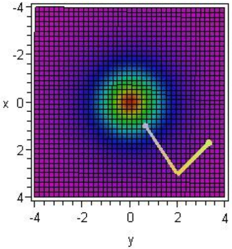
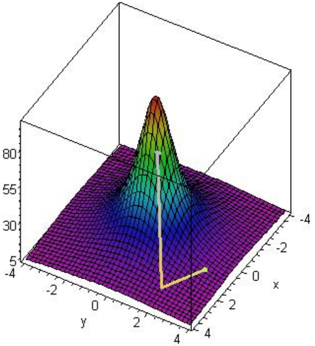
$$\frac{1}{\sqrt{13}}\langle -2, 3\rangle$$


-4
-2
x 0
2
4
-4
-2
0
2
4
y


80
55
30
5
-4
-2
0
y
2
4
-4
-2
0
x
2
4

Extending the Definition

Let $w = f(x, y, z)$. Then

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

and the directional derivative:

$$D_{\vec{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}$$

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Which is:

$$\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$$

for any curve on the surface.

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$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0$$

Which is:

$$\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$$

for any curve on the surface.Therefore, the gradient vector is the **normal vector for the tangent plane**.

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Consider a level surface, $F(x, y, z) = k$.Thinking of a curve C on the surface, we can take x, y, z as functions of t (they form the parametric functions for C).Then:

$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0$$

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$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

How is this related to our usual tangent plane?

We usually have $z = f(x, y)$. Rewriting as a level surface,

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becomes:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is our usual definition.

Read over the normal line to surface S at point $P(x_0, y_0, z_0)$. This line goes through (x_0, y_0, z_0) in the direction of the gradient, $\nabla F(x_0, y_0, z_0)$.