Exam 2 Sample SOLUTIONS

1. True or False, and explain:

(a) There exists a function f with continuous second partial derivatives such that

$$f_x(x,y) = x + y^2 \qquad f_y = x - y^2$$

SOLUTION: False. If the function has continuous second partial derivatives, then Clairaut's Theorem would apply (and $f_{xy} = f_{yx}$). However, in this case:

$$f_{xy} = 2y \qquad f_{yx} = -2y$$

(b) The function f below is continuous at the origin.

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

SOLUTION: Check the limit- First, how about $y = x$ versus $y = -x$?

$$\lim_{(x,x)\to(0,0)} \frac{2x^2}{3x^2} = \frac{2}{3} \qquad \lim_{(x,-x)\to(0,0)} \frac{-2x^2}{3x^2} = \frac{-2}{3}$$

Yep, that did it- The limit does not exist at the origin, therefore the function is not continuous at the origin (it is continuous at all other points in the domain).

(c) If $\vec{r}(t)$ is a differentiable vector function, then

$$\frac{d}{dt}|\vec{r}(t)| = |\vec{r}\,'(t)|$$

SOLUTION: False.

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{2}(\mathbf{r}(t)\cdot\mathbf{r}(t))^{-1/2}\left(\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t)\right) = \frac{\mathbf{r}'(t)\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|}$$

Extra: If you're not sure about it, try to verify the formula with $\mathbf{r}(t) = \langle 3t^2, 6t-5\rangle$.

(d) If $z = 1 - x^2 - y^2$, then the linearization of z at $(1,1)$ is

$$L(x,y) = -2x(x-1) - 2y(y-1)$$

SOLUTION: False for two reasons. We have forgotten to evaluate the partial derivatives of f at the base point $(1,1)$ (and so the resulting formula is not linear). We have also forgotten to evaluate the function itself at $(1,1)$. The linearization should be:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = -1 - 2(x-1) - 2(y-1)$$

(e) We can always use the theorem: $\nabla f(a,b)\cdot\vec{u}$ to compute the directional derivative at (a,b) in the direction of $\vec{u}$.
SOLUTION: False. This formula only works if f is differentiable at (a,b) (See Exercise 3 below).

(f) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve.
SOLUTION: False. The magnitude of $\mathbf{r}(t)$ is the velocity. For example, $\mathbf{r}(3t)$ will have a magnitude that is three times that of the original- If you want an actual example, consider

$$\mathbf{r}(t) = \langle\cos(t), \sin(t)\rangle$$

At the point on the unit circle $(1/\sqrt{2}, 1/\sqrt{2})$, the magnitude of $\mathbf{r}'(\pi/4) = 1$. Replace t by $3t$ (and evaluate at $t = \pi/12$ to have the same point on the curve), and the speed is 3 instead of 1.

Why did we bring this up? If we re-parameterize with respect to *arc length*, the velocity is always 1 unit (so at $s = 1$, you've travelled one unit of length, etc).

(g) If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}\,'(t)$$

SOLUTION: False. It looks like the product rule:

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}\,'(t)$$

(h) If $f_x(a,b)$ and $f_y(a,b)$ both exist, then f is differentiable at (a,b).
SOLUTION: False. Our theorem says that in order to conclude that f is differentiable at (a,b), the partial derivatives must be *continuous* at (a,b). Just having the partial derivatives exist at a point is a weak condition- It is not enough to even have continuity.

(i) At a given point on a curve $(x(t_0), y(t_0), z_0(t))$, the osculating plane through that point is the plane through $(x(t_0), y(t_0), z(t_0))$ with normal vector is $\vec{B}(t_0)$.
SOLUTION: True (by definition).

2. Show that, if $|\vec{r}(t)|$ is a constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$. (HINT: Differentiate $|\vec{r}(t)|^2 = k$)

SOLUTION: Using the hint,

$$0 = \frac{d}{dt}k = \frac{d}{dt}\left(|\vec{r}(t)|^2\right) = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \vec{r}\,'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}\,'(t) = 2\vec{r}\,'(t)\cdot\vec{r}(t)$$

Therefore, the dot product is zero (and so $\vec{r}\,'(t)$ and $\vec{r}(t)$ are orthogonal).

3. Is it possible for the directional derivative to exist for every unit vector $\vec{u}$ at some point (a,b), but f is still not differentiable there?

Consider the function $f(x,y) = \sqrt[3]{x^2y}$. Show that the directional derivative exists at the origin (by letting $\vec{u} = \langle\cos(\theta), \sin(\theta)\rangle$ and using the **definition**), BUT, f is not differentiable at the origin (because if it were, we could use $\nabla f\cdot\vec{u}$ to compute $D_{\vec{u}}f$).

SOLUTION: Compute the directional derivative at the origin by using the definition:

$$D_uf(0,0) = \lim_{h\to 0}\frac{f(0+h\cos(\theta), 0+h\sin(\theta)) - f(0,0)}{h} = \lim_{h\to 0}\frac{h\sqrt[3]{\cos^2(\theta)\sin(\theta)}}{h} = \sqrt[3]{\cos^2(\theta)\sin(\theta)}$$

Notice that by using the definition, $\theta = 0$ corresponds to the rate of change parallel to the $x-$axis, and $\theta = \pi/2$ is the rate of change parallel to the $y-$axis:

$$f_x(0,0) = 0 \qquad f_y(0,0) = 0$$

so that, if f were differentiable at the origin, we could use

$$D_uf = \nabla f\cdot\vec{u} = 0$$

for every vector $\vec{u}$, but that is not the case (our directional derivative is not always zero).

4. If $f(x,y) = \sin(2x+3y)$, then find the linearization of f at $(-3,2)$.

SOLUTION: We have $f(-3,2) = \sin(0) = 0$ and

$$f_x(x,y) = 2\cos(2x+3y) \quad\Rightarrow\quad f_x(-3,2) = 2$$

$$f_y(x,y) = 3\cos(2x+3y) \quad\Rightarrow\quad f_y(-3,2) = 3$$

Therefore,

$$L(x,y) = 0 + 2(x+3) + 3(y-2) = 2(x+3) + 3(y-2)$$

5. The radius of a right circular cone is increasing at a rate of 3.5 inches per second while its height is decreasing at a rate of 4.3 inches per second. At what rate is the volume changing when the radius is 160 inches and the height is 200 inches? ($V = \frac{1}{3}\pi r^2 h$)

   SOLUTION:
$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \, \frac{dh}{dt}$$
   Use $r = 160$, $h = 200$, $dr/dt = 3.5$ and $dh/dt = -4.3$, $dV/dt \approx 37973.3\pi$

6. Find the differential of the function: $v = y\cos(xy)$

   SOLUTION:
$$dv = v_x\,dx + v_y\,dy = (-y^2\sin(xy)\,dx + (\cos(xy) - xy\sin(xy))\,dy$$

7. Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$, and the direction in which it occurs.

   SOLUTION: The maximum rate of change occurs if we move in the direction of the gradient. We see this by recalling that:
$$D_u f = \nabla f \cdot \vec{u} = |\nabla f|\cos(\theta)$$
   so we find the gradient at the point $(2, 1)$
$$\nabla f = \langle 4, 9/2\rangle$$
   So if we move in that direction, the we get the max rate of change, which is
$$|\nabla f| = \sqrt{4^2 + \frac{81}{4}} = \frac{\sqrt{145}}{2} \approx 6.02$$

8. Find an expression for $\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right]$

   SOLUTION:
$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right] = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})'$$
   Taking this derivative, we see that
$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right] = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u}\,(\mathbf{v}' \times \mathbf{w} + \mathbf{v} \times \mathbf{w}')$$

9. Use Lagrange Multipliers to find the maximum and minimum of f subject to the given constraints:
$$f(x, y) = x^2y \qquad x^2 + y^2 = 1$$
   Set up the equations by first computing the gradients of f and g:
$$\nabla f = \langle 2xy, x^2\rangle \qquad \nabla g = \langle 2x, 2y\rangle$$
   So the system of equations is given below. Solve that (recall that $\lambda \neq 0$) for our candidates:
$$\begin{aligned} 2xy &= 2\lambda x \\ x^2 &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$
   If $x \neq 0$ in the first equation, then $\lambda = y$. Going to the second equation, that implies that $x^2 = 2y^2$. Now to the third equation, we can solve for y, and therefore also x:
$$3y^2 = 1 \quad \Rightarrow \quad y = \pm\sqrt{\frac{1}{3}} \quad \Rightarrow \quad x = \pm\sqrt{\frac{2}{3}}$$

Are there any other solutions? If $x = 0$ in the first equation, then $y = \pm 1$ from the third equation, but that would imply that λ must be zero (and it shouldn't be).

Substitute into f to find the max/min:

$$f\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \frac{2}{3\sqrt{3}} \qquad f\left(\pm\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right) = \frac{-2}{3\sqrt{3}}$$

10. The curves below intersect at the origin. Find the angle of intersection to the nearest degree:

$$\vec{r}_1(t) = \langle t, t^2, t^9\rangle \quad \vec{r}_2(t) = \langle \sin(t), \sin(5t), t\rangle$$

The angle of intersection is the angle between the tangent vectors at the origin. First differentiate, then evaluate at $t = 0$:

$$\vec{r}_1{}'(t) = \langle 1, 2t, 9t^8\rangle \quad \vec{r}_2{}'(t) = \langle \cos(t), 5\cos(5t), 1\rangle \quad \Rightarrow \quad \langle 1, 0, 0\rangle, \langle 1, 5, 1\rangle$$

To find the angle, we use the relationship:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta)$$

In our case:

$$\cos(\theta) = \frac{1}{\sqrt{1^2 + 5^2 + 1^2}} \quad \Rightarrow \quad \theta = \cos^{-1}(1/\sqrt{27}) \approx 79^\circ$$

11. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

SOLUTION: Given any point in three dimensions, x, y, z, the distance to the origin is $\sqrt{a^2 + b^2 + c^2}$. If we are only minimizing the distance, we can work with the squared distance- Therefore, we re-formulate the question as:

"Find the minimum of $x^2 + y^2 + z^2$ such that $xy^2z^3 = 2$", which is the method of Lagrange Multipliers. First find the gradients of $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = xy^2z^3$, and make them parallel:

$$\nabla f = \langle 2x, 2y, 2z\rangle \qquad \nabla g = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$$

Now we have four equations and four unknowns. Notice that the constraint $g(x, y, z) = 2$ implies that none of x, y, z or λ can be zero, so we can feel free to divide by any of them. In each case, we solve for λ and substitute:

$$\begin{array}{rlll} 2x & = \lambda y^2 z^3 & & \\ 2y & = 2\lambda xyz^3 & \text{Eqns 1, 2} \Rightarrow & y^2 = 2x^2 \\ 2z & = 3\lambda xy^2z^2 & \text{Eqns 1, 3} \Rightarrow & z^2 = 3x^2 \quad z = \pm\sqrt{3}x \\ xy^2z^3 & = 2 & & \end{array}$$

Put these back into Equation 4, and we have:

$$x(2x^2)(\pm\sqrt{3}x)^3 = 2 \quad \Rightarrow \quad x^6 = \frac{2}{2 \cdot \pm 3^{3/2}} \quad \Rightarrow \quad x = 3^{-1/4}$$

From $z = \pm\sqrt{3}x$ and Equation 1, we see that x and z must have the same sign, so $x = \pm 3^{-1/4}$, $z = \pm 3^{1/4}$ (respectively), and coming around to y, we have four points:

$$f\left(\pm 3^{-1/4}, 2^{1/2}3^{-1/4}, \pm 3^{1/4}\right) = 2\sqrt{3} \qquad f\left(\pm 3^{-1/4}, -2^{1/2}3^{-1/4}, \pm 3^{1/4}\right) = 2\sqrt{3}$$

There is no maximum (the surface is not bounded). We can see what g looks like in the figure below.

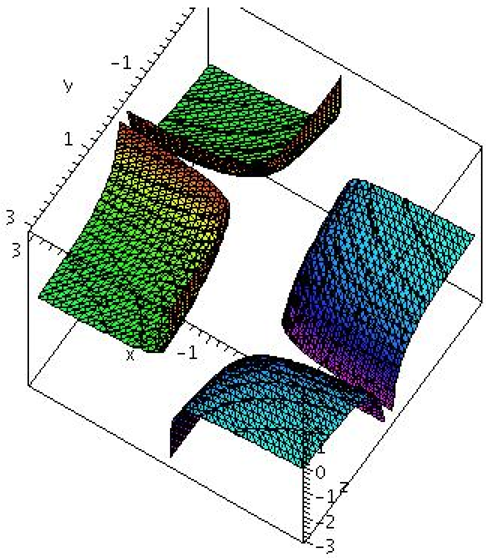

Figure 1: Plot of the (implicit) surface $xy^2z^3 = 2$. We can see that there are four points that are closest to the origin (global min), but no global maximum (the surface is not bounded).

12. Find the equation of the tangent plane and normal line to the given surface at the specified point:

$$x^2 + 2y^2 - 3z^2 = 3 \qquad (2, -1, 1)$$

SOLUTION: This is an implicitly defined surface of the form $F(x, y, z) = k$, therefore, we know that ∇F is orthogonal to the tangent planes on the surface. Compute ∇F at $(2, -1, 1)$, and construct the plane and line:

$$F_x = 2x \quad F_y = 4y \quad F_z = -6z \quad \Rightarrow \quad \nabla F(2, -1, 1) = \langle 4, -4, -6 \rangle$$

Thus, the tangent plane is:

$$4(x - 2) - 4(y + 1) - 6(z - 1) = 0$$

The normal line goes in the direction of the gradient, starting at the given point. In parametric form,

$$x(t) = 2 + 4t \quad y(t) = -1 - 4t \quad z(t) = 1 - 6t$$

13. If $z = x^2 - y^2$, $x = w + 4t$, $y = w^2 - 5t + 4$, $w = r^2 - 5u$, $t = 3r + 5u$, find $\partial z/\partial r$.

See hand-written solution, attached.

14. If $x^2 + y^2 + z^2 = 3xyz$ and we treat z as an implicit function of z, then find $\partial z/\partial x$ and $\partial z/\partial y$.

See hand-written solution, attached.

15. Short Answer:

(a) If the velocity is $\sin(t)\vec{i} - \cos(t)\vec{j} + 14t\vec{k}$, and the initial position is $\vec{i} + \vec{j} + 2\vec{k}$, (i) Find the position function, (ii) Find the acceleration, (iii) Find the tangential and normal components of acceleration.

SOLUTION: The third part gets very messy (Sorry! - That question was just meant for you to look up and compute with those formulas):

(i) $\vec{r}(t) = \langle \cos(t), 1 - \sin(t), 7t^2 + 2 \rangle$

(ii) $\vec{a}(t) = \langle \cos(t), -\sin(t), 14 \rangle$

(iii)

$$a_T = \frac{\langle \sin(t), -\cos(t), 14t\rangle \cdot \langle \cos(t), -\sin(t), 14\rangle}{\sqrt{1+14t^2}}$$

$$a_N = \frac{|\langle \sin(t), -\cos(t), 14t\rangle \times \langle \cos(t), -\sin(t), 14\rangle|}{\sqrt{1+14t^2}}$$

(b) Find the length of the curve:

$$\mathbf{r}(t) = \langle 9\sin(t), 3t, 9\cos(t)\rangle, -\pi \le t \le 2\pi$$

SOLUTION:

$$\int_{-\pi}^{2\pi} |\vec{r}\,'(t)|\, dt = \int_{-\pi}^{2\pi} \sqrt{81\cos^2(t) + 9 + 81\sin^2(t)}\, dt = 9\sqrt{10}\pi$$

16. At what point does $y = 8e^x$ have maximum curvature?

SOLUTION: The curvature is given by (see the formulas on the question page)

$$\kappa = \frac{8e^x}{[1+(8e^x)^2]^{3/2}}$$

Take the derivative, set it to zero to find CP's:

$$\kappa'(x) = -\frac{8e^x(-1+128e^{2x})}{[1+64e^{2x}]^{5/2}} = 0 \Rightarrow \quad 128e^{2x} = 1$$

From which we get the solution, $x = -\frac{7}{2}\ln(2) \approx -2.43$. To see if it is a max, use the first derivative test and see that the derivative is going from positive to negative.

17. Find the equation of the normal line through the level curve $4 = \sqrt{5x-4y}$ at $(4,1)$ using a gradient.

SOLUTION: The gradient of g is orthogonal to its level curve at $\sqrt{5x-4y} = 4$. Find the gradient of g at $(4,1)$. We should find that it is:

$$\nabla g = \frac{1}{8}\langle 5, -4\rangle$$

Therefore, the line (in parametric and symmetric form) is:

$$x(t) = 4+5t \quad y(t) = 1-4t \quad \text{or} \quad \frac{x-4}{5} = \frac{y-1}{-4}$$

Notice that the slope is $-4/5$. If we wanted to check our answer, we could find the slope of the tangent line:

$$5x-4y = 16 \quad \Rightarrow \quad 5 - 4\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{5}{4}$$

18. Find all points at which the direction of fastest change in the function $f(x,y) = x^2+y^2-2x-4y$ is $\vec{i}+\vec{j}$.

SOLUTION: The direction of the fastest increase is in the direction of the gradient. Therefore, another way to phrase this question is: When is the gradient pointing in the direction of $\langle 1,1\rangle$ (very reminiscent of the Lagrange Multiplier):

$$\nabla f = k\langle 1,1\rangle \quad \Rightarrow \quad \langle 2x-2, 2y-4\rangle = \langle k, k\rangle$$

So $k = 2x-2$ and $k = 2y-4$, therefore, the points are on the line $2x-2 = 2y-4$, or $y = x+1$.

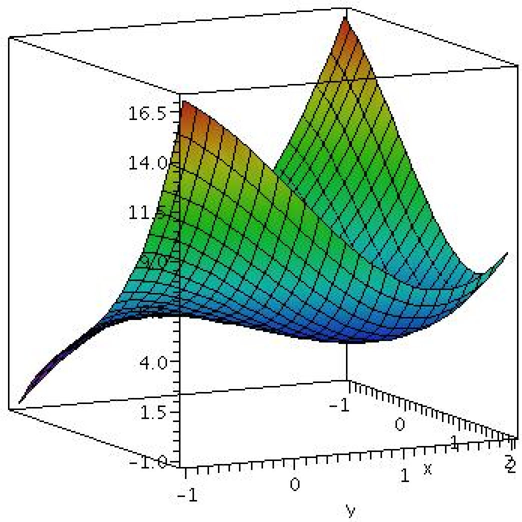

Figure 2: Figure for the last problem.

19. Find and classify the critical points:

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$

SOLUTION: The partial derivatives are:

$$f_x = 3x^2 - 3y \quad f_y = 3y^2 - 3x \quad f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$$

The critical points are where $x^2 = y$ and $y^2 = x$, so x, y are both zero or positive:

$$x^4 = x \quad \Rightarrow \quad x^4 - x = 0 \quad \Rightarrow \quad x(x^3 - 1) = 0$$

so $x = 0, y = 0$ or $x = 1, y = 1$. Put these points into the Second Derivatives Test:

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = -9 < 0 \quad \Rightarrow \quad \text{The origin is a SADDLE}$$

$$f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) = 36 - 9 > 0 \quad f_{xx}(1,1) > 0 \Rightarrow \quad \text{Local MIN}$$