Random Questions: Calc III

For the final exam, you may bring a 3" $\times$ 5" card of notes (both sides) with you. You should bring a calculator. To study, please be sure to look over the old exams, old quizzes, then you might look at a homework problem or two over the sections that you may be fuzzy on.

1. Find the limit, if it exists:

$$\lim_{x\to 2}\frac{x^2-6x+8}{x-2} \qquad \lim_{(x,y)\to(0,0)}\frac{6x^3y}{2x^4+y^4} \qquad \lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

   What is the most salient difference between the first limit and the other two?

2. Find the projection of the vector $\langle 1,4,6\rangle$ onto the vector $\langle -2,5,-1\rangle$. For what vector would the projection have the smallest magnitude? The largest magnitude?

3. Find the local maximum and minimum values and saddle point(s) of the function: $f(x,y)=x^3y+12x^2-8y$.

4. Same function as in 3, but find the global maximum if $-1\le x\le 1$ and $-1\le y\le 1$.

5. Suppose E is the region inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

   (a) Find the volume using an appropriate triple integral (Yes, it is easy to find geometrically, so verify your answer!).

   (b) The sides of the cylinder can be written parameterically as:

$$\vec{r}(\theta,z)=\langle 4\cos(\theta),4\sin(\theta),z\rangle$$

   Write the integral(s) for the surface area (Yes, it is easy to find geometrically- Verify your answer!)

6. Find the area of the parallelogram formed by the vectors $\langle 6,3,-1\rangle$, $\langle 0,1,2\rangle$. Find the volume of the parallelepiped if we add a third vector, $\langle 4,-2,5\rangle$

7. Is a function differentiable if the partial derivatives both exist at a point? Before you answer, consider the following example:

   Let $f(x)=x^{1/3}y^{1/3}$

   (a) If $x\neq 0$, compute $f_x(x,y)$ (similarly for $f_y(x,y)$).

   (b) Use the **definition** of $f_x(0,0)$ to show that the partial derivative at $(0,0)$ is zero (similarly, show it for $f_y(0,0)$.

   The graph of f would show you that it is not locally linear at the origin.

8. If the partial derivatives for a function exist at a point, does that mean that the function is continuous there? Before you answer, consider the following example:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) Show that f is not continuous at the origin.

   (b) Show, using the definition, that $f_x(0,0) = 0$, and $f_y(0,0) = 0$

   (NOTE: Since f is not continuous at $(0,0)$, it is also not differentiable at $(0,0)$ even though the partial derivatives exist there).

9. Our **definition** of differentiability is the textbook's Theorem 8 (14.4): If f_x and f_y are *continuous* on a small disk containing the point (a,b), then f is differentiable at (a,b).

   Show that $f(x) = x^{1/3}y^{1/3}$ is not differentiable at the origin.

10. True or False?

    (a) If f is differentiable at (a,b) then f is continuous at (a,b).

    (b) If f is not continuous at (a,b), then f cannot be differentiable at (a,b).

    (c) If f is not continuous at (a,b), then f_x and/or f_y cannot exist at (a,b).

11. If $z = x^2 - xy + 3y^2$, and (x,y) changes from $(3,-1)$ to $(2.96,-0.95)$, compare the values of Δz and dz.

12. Find the equation of the tangent plane to $z = \frac{2x+3}{4y+1}$ at $(0,0)$. Would this be the same thing as linearization?

13. If $u = \sqrt{r^2+s^2}$, $r = y + x\cos(t)$ and $s = x + y\sin(t)$, compute $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial t$ when $x = 1$, $y = 2$ and $t = 0$.

14. Show that the direction in which the rate of change of f is greatest is in the direction of the gradient. You should start with:

$$D_{\vec{u}}f(x,y,z) = \nabla f \cdot \vec{u}$$

    What *is* the greatest rate of change of f if you go in that direction?

    Illustrate your answer with the following example: $f(x,y,z) = 5x^2 - 3xy + xyz$ at the point $P(3,4,5)$.

15. Let $yz = \ln(x+z)$. Find the equations of the tangent plane and normal line to the surface at $(0,0,1)$.

16. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line and the gradient vector.

17. Let the curve C be defined parametrically by: $x = t^2$ and $y = t^4 - 1$. Find the equation of the tangent line at $(4, 15)$.

18. Find the work:

    (a) of the vector field $\vec{F} = \langle x, -z, y\rangle$ acting on a particle along the path $\vec{r}(t) = \langle 2t, 3t, -t^2\rangle$, for $-1 \leq t \leq 1$.

    (b) of the constant force $\vec{F} = \langle 8, -6, 9\rangle$ that moves an object from the point $(0, 10, 8)$ to $(6, 12, 20)$ along a straight line.

    (c) of the vector field $\vec{F} = \langle 3y - \mathrm{e}^{\sin(x)}, 7x + \sqrt{y^4 + 1}\rangle$ on a particle going around the curve C, which in this case is a circle of radius 3 (assume CCW).

    (d) of the vector field $\vec{F} = \langle -y^2, x, z^2\rangle$, and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ (C is CCW from above).

19. A region E is a tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 2)$, $(0, 1, 0)$ and $(1, 1/2, 0)$.

    (a) Find the three planes representing the three faces of E.

    (b) Find six integrals that would give the volume of E. (NOTE: Careful in looking at the projection into the yz plane- there are actually two regions to consider).

20. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 y$ subject to the constraint $x^2 + 2y^2 = 6$.

21. Set up an integral to determine the arc length of one period of the sine function (do not evaluate).

22. Evaluate

    $$\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$$

    where C is the line segment from $(1, 1, 1)$ to $(1, 2, 4)$.

    Repeat the last problem, except C is the curve $\vec{r}(t) = \langle 1, t, t^2\rangle$ for $1 \leq t \leq 2$.

23. Find the flux across the surface:

    $$\vec{F} = \langle xy, yz, zx\rangle$$

    where the surface is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

24. Same vector field as Exercise 21, but now let the surface be the edges on the cube: $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$.