Exam 1 Sample SOLUTIONS

Be sure to look over your old quizzes and homework as well. For more review questions, see the Chapter 12 Review, pg. 812-813.

1. Eliminate the parameter to find a Cartesian equation for the curve: $x = \mathrm{e}^t$, $y = 2\mathrm{e}^t$.

   SOLUTION: $y = 2x$

   Find the arc length of the curve from $t = 1$ to $t = 2$ of the original parametric form, then for the (x, y) form.

   SOLUTION:

$$L = \int_{t=1}^{t=2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{t=1}^{t=2} \sqrt{5}\mathrm{e}^t\, dt = \sqrt{5}(\mathrm{e}^2 - \mathrm{e})$$

$$L = \int_{x=\mathrm{e}^1}^{x=\mathrm{e}^2} \sqrt{1 + (f'(x))^2}\, dx = \int_{\mathrm{e}}^{\mathrm{e}^2} \sqrt{1+4}\, dx = \sqrt{5}(\mathrm{e}^2 - \mathrm{e})$$

2. Find dy/dx and d^2y/dx^2, if $x = t^3 - 12t$ and $y = t^2 - 1$

   SOLUTION:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 12}$$

$$\frac{d^2y}{dx^2} = \frac{d/dt(dy/dx)}{dx/dt} = \frac{\frac{-6t^2-24}{(3t^2-12)^2}}{(3t^2-12)} = \frac{-6t^2 - 24}{(3t^2-12)^3}$$

3. A plane is given as $2x + 3y + 4z = 12$. Sketch it in the first octant by drawing the lines of intersection between it and the coordinate planes.

   Give the equations of the lines of intersection that you have drawn (in parametric form).

   SOLUTION: The line of intersection with the xy plane is found by setting $z = 0$:

$$2x + 3y = 12, z = 0$$

   There are multiple ways of writing this in parametric form. One way is to form points P, Q, R from the coordinate axes:

$$P(6, 0, 0) \quad Q(0, 4, 0) \quad R(0, 0, 3)$$

   The first line goes through $(6, 0, 0)$ in the direction $\overrightarrow{QP}$:

$$x = 6 - 6t \quad y = 0 \quad z = 3t$$

   The second line goes through $(0, 0, 3)$ in the direction $\overrightarrow{QR}$:

$$x = 0 \quad y = 4t \quad z = 3 - 3t$$

   The third line goes through $(0, 4, 0)$ in the direction $\overrightarrow{RP}$:

$$x = 6t \quad y = 4 - 4t \quad z = 0$$

4. Convert the polar equation to Cartesian: $r = \tan(\theta)\sec(\theta)$

   SOLUTION: We might re-write using sines and cosines: for x, y:

$$r = \frac{\sin(\theta)}{\cos^2(\theta)} \qquad \Rightarrow \qquad r\cos^2(\theta) = \sin(\theta)$$

   so that one way to simplify is to multiply both sides by r and do the conversion:

$$r^2\cos^2(\theta) = r\sin(\theta) \qquad \Rightarrow \qquad x^2 = y$$

5. Convert the equation from Cartesian to polar of the form $r = f(\theta)$: $xy = 4$

   SOLUTION: Substitute directly and isolate r:

$$r^2\cos(\theta)\sin(\theta) = 4 \qquad \Rightarrow \qquad r^2 = \frac{4\sin(\theta)}{\cos(\theta)}$$

   (Sorry- this probably shouldn't be expressed as $r = f(\theta)$- Fine to leave your answer here).

6. Find the area of the surface obtained by rotating the curve about the $x-$axis:

$$x = 3t - t^3 \qquad y = 3t^2 \qquad 0 \le t \le 1$$

   SOLUTIONS:

   NOTE: You should think of this as a generic curve given in parametric form- You may assume it can be written as $y = f(x)$-

$$\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^1 3t^2\sqrt{9t^4 + 18t^2 + 9}\,dt =$$

$$18\pi\int_0^1 t^2\sqrt{(t^2+1)^2}\,dt = 18\pi\int_0^1 t^2(t^2+1)\,dt = \frac{48}{5}\pi$$

   You may have had some problems with the algebra- That's OK, as long as the big picture/formula are clear.

7. Find the slope of the tangent line to the given polar curve at the point specified by θ:

$$r = 2 - \sin(\theta) \qquad \theta = \frac{\pi}{3}$$

   SOLUTION:

$$x = r\cos(\theta) = (2 - \sin(\theta))\cos(\theta) = 2\cos(\theta) - \sin(\theta)\cos(\theta)$$

   and

$$y = r\sin(\theta) = (2 - \sin(\theta)\sin(\theta) = 2\sin(\theta) - \sin^2(\theta)$$

   After some algebra and trig,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$$

8. Find the distance between the point $(1, 2, 3)$ and the plane $x + y + z = 1$.

   SOLUTION: Distance is:

$$\frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1 + 2 + 3 - 1|}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

9. Find the distance between the planes $x + y + z = 1$ and $3x + 3y + 3z - 5 = 0$.

   SOLUTION: Do a quick check that they are parallel. You may use a shortcut formula from the exercises, or take any point from one plane, and find the distance to the second. For example, $(1, 0, 0)$ is on the first plane. The distance to the second is:

$$\frac{|3 - 5|}{\sqrt{3^2 + 3^2 + 3^2}} = \frac{2}{\sqrt{27}} = \frac{2}{3\sqrt{3}}$$

10. Find the angle between the vectors $\langle 1, 2, 3\rangle$ and $\langle 1, 0, 1\rangle$. Find the projection of the first vector onto the second.

    SOLUTION:

$$\cos(\theta) = \frac{1 + 0 + 3}{\sqrt{1 + 4 + 9}\sqrt{1 + 1}} = \frac{4}{\sqrt{28}} = \frac{2}{\sqrt{7}}$$

    Taking the inverse cosine give about 40.9 degrees.

11. If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° with the horizontal, find the work in pulling the sled 10 meters.

    SOLUTION:

$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos(\theta) = 50 \cdot 10 \cdot \cos(38) \approx 394$$

12. Graphical problems: For example p. 627, and Ex. 49-50, p. 648, and p. 777.

13. Do the lines below intersect? If so, find the point of intersection. If not, find the distance between them.

$$x = 3 + t, y = 2 - t, z = 1 \qquad x = 2 + s, y = 1 + 2s, z = 2 - s$$

    SOLUTION: To check, set them equal to each other. I like to use the two easiest equations first- For example, from the equations for x, we have $t = s - 1$, so in the second equation,

$$2 - (s - 1) = 1 + 2s \quad \Rightarrow \quad s = \frac{2}{3} \quad \Rightarrow \quad t = -\frac{1}{3}$$

    Substituting into the third equation, we see that either $z = 1$ or $z = 4/3$, therefore the lines do not intersect. Furthermore, they are not parallel since $\langle 1, -1, 0\rangle$ is not a multiple of $\langle 1, 2, -1\rangle$- Therefore, the two lines are skew.

    To find the distance between the lines, we find parallel planes containing the two lines, then find the distance between them. The vector that is normal to the plane should be perpendicular to the two lines as well, so taking the cross product of the two direction vectors we get:

$$\vec{n} = \langle 1, -1, 0\rangle \times \langle 1, 2, -1\rangle = \langle 1, 1, 3\rangle$$

We need a point for each plane. For example, we might use $(3,2,1)$ and $(2,1,2)$, respectively. The (simplified) equations for the two planes are:

$$x + y + 3z - 8 = 0 \qquad x + y + 3z - 9 = 0$$

You can take a point on the first plane and find the distance to the second. For example, $(3,2,1)$ as we had before. Then the distance is:

$$\frac{|3 + 2 + 3(1) - 9|}{\sqrt{11}} = \frac{1}{\sqrt{11}}$$

14. Find an equation for the surface obtained by rotating the

    parabola $y = x^2$ about the $y-$axis. (Hint: In 3-d, if you fix a $y-$value, what shape should you have in the $xz-$plane?)

    SOLUTION: We can think of the graph $y = x^2$ as the result of plotting the trace when $z = 0$. If we rotate the curve about the $y-$axis, we'll get something like a rounded vase- If we take a cross section parallel to the $xz-$plane when $y = k$, we should get a circle of radius $\sqrt{k}$, so our equation is:

    $$x^2 + z^2 = y$$

15. Write the parametric equations for each situation:

    (a) Go from $(1,2)$ to $(-3,2)$ as time runs from 0 to 1.
    SOLUTION: Recall that if we want to go from a to b, then use $a(1-t) + bt$

    $$\begin{array}{lll} x & = 1(1-t) - 3t & = 1 - 4t \\ y & = 2(1-t) - 2t & = 2 \end{array}$$

    (b) Go around the unit circle twice in clockwise fashion starting at $(-1,0)$ as $0 \le t \le 1$.
    SOLUTION: One way to construct this is to start at $(1,0)$ and move clockwise by reversing t. Because cosine is even and sine is odd, we have:

    $$\langle \cos(-t), \sin(-t) \rangle = \langle \cos(t), -\sin(t) \rangle$$

    Next, shift time back by π by taking:

    $$\langle \cos(t + \pi), -\sin(t + \pi) \rangle$$

    Notice now at $t = 0$, we are at $(-1,0)$. At $t = \pi/2$, we are at $(0,-1)$ (that's the right direction of travel). Rather than time running from 0 to 4π, we want time to run from 0 to 1- You can think of this as a change of variables,

    $$t = 4\pi s$$

    And substitute, getting our final answer parametrized in s:

    $$x = \cos(4\pi s + \pi), y = -\sin(4\pi s + \pi)$$

(c) Follow the curve $y = 3x^2 + 5x + 2$ as $-\infty < t < \infty$.
SOLUTION: The simplest conversion is to take $x = t$:

$$\begin{aligned} x &= t \\ y &= 3t^2 + 5t + 2 \end{aligned}$$

(d) Follow the curve $r = 1+2\cos(\theta)$ (in the Cartesian coordinate system) as $0 \le \theta \le 2\pi$.
SOLUTION: We know that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so just substitute:

$$\begin{aligned} x &= (1 + 2\cos(\theta))\cos(\theta) \\ y &= (1 + 2\cos(\theta))\sin(\theta) \end{aligned}$$

16. Find the error (if there is one): Let $\mathbf{u}$ be a vector such that $|\mathbf{u}| = \sqrt{2}$. Choose a vector $\mathbf{v} \neq \mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 2$. Now we have:

$$\begin{aligned} 2(\mathbf{u}\cdot\mathbf{u}) &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} \\ 2(\mathbf{u}\cdot\mathbf{u}) - 2(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - 2(\mathbf{u}\cdot\mathbf{v}) \\ 2(\mathbf{u}\cdot\mathbf{u}) - 2(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} \\ 2\mathbf{u}\cdot(\mathbf{u} - \mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u} - \mathbf{v}) \\ 2\mathbf{u} &= \mathbf{u} \\ 2 &= 1 \end{aligned}$$

SOLUTION:

We'll recall that $\mathbf{a}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{b}$ does NOT imply that $\mathbf{a} = \mathbf{c}$. In this case, you might note that the second, third and fourth lines are all of the form "zero=zero". Until the 5th line!

17. Show that $r = 2\cos(\theta)$ is a circle by converting it into Cartesian coordinates and writing it in standard form.

SOLUTION: Idea is not only the conversion, but also completing the square:

$$r^2 = r\cos(\theta) \quad \Rightarrow \quad x^2 + y^2 = x \quad \Rightarrow \quad x^2 - x + y^2 = 0 \quad \Rightarrow \quad (x - 1/2)^2 + y^2 = 1/4$$

18. Find the scalar and vector projections of $\mathbf{a}$ onto $\mathbf{b}$, if

- $\mathbf{a} = \langle -2, 3, -6\rangle \qquad \mathbf{b} = \langle 5, -1, 4\rangle$
  SOLUTION: The scalar projection is what the text refers to as

  $$\mathrm{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{-10 - 3 - 24}{\sqrt{25 + 1 + 16}} = \frac{-37}{\sqrt{42}}$$

  The vector projection is then easy to compute:

  $$\mathrm{Proj}_{\mathbf{b}}(\mathbf{a}) = \frac{-37}{\sqrt{42}}\left(\frac{1}{\sqrt{42}}\right)\langle 5, -1, 4\rangle = \frac{-37}{42}\langle 5, -1, 4\rangle$$

- $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$
  SOLUTION: This one was included so that you would have practice with the standard basis vectors $\vec{i}, \vec{j}, \vec{k}$. In this case, the scalar projection is $-1/\sqrt{2}$ and the projection is $(1/2)\mathbf{b} = (1/2)\langle 0, 1, -1\rangle$.

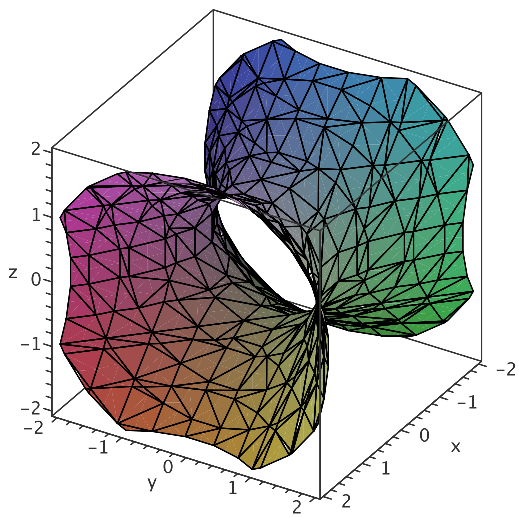

Figure 1: Solution to Exercise 19.

19. Sketch a few traces of the surface $y^2 + z^2 = 1 + x^2$ and describe the resulting surface (in words and/or as a sketch in three-dimensions).

    SOLUTION: See Figure 1.

20. Let a line L and a plane P be defined as:

$$L: \quad x = \frac{y+2}{3} = -z \qquad P_1: \quad x + y + z = 1$$

NOTE: As part of the solution, it may be easiest to deal with the line in parametric form- See if you can convert it to:

$$x = t \quad y = -2 + 3t \quad z = -t$$

(a) Find the point of intersection between the plane and the line.
SOLUTION: Use the parametric form of the line, substitute it into the equation of the plane to get that $t = 1$. To check, substitute $t = 1$ into the equation of the line to get $(1, 1, -1)$. Substitute into the equation of a plane to see that the equation is satisfied.

(b) Given the point $Q(1, 2, 1)$, find the plane P_2 that contains the line L and the point Q.
SOLUTION: We have lots of points in the plane- We need the normal vector. We should see that the normal vector will be orthogonal to the line, or

$$\vec{n} \perp \langle 1, 3, -1 \rangle$$

Taking a vector from the line to the point (for example, from $(1, 2, 1)$ to $(0, -2, 0)$), we have that

$$\vec{n} \perp \langle 1, 4, 1 \rangle$$

Take the cross product to find that $\vec{n} = \langle 7, -2, 1 \rangle$, so one form of the plane is:

$$7(x - 1) - 2(y - 2) + (z - 1) = 0$$

(c) Find the distance between Q and the plane P_1. What is the distance between planes P_1, P_2?
SOLUTION: Use the distance formula as before:

$$\frac{|1+2+1-1|}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Planes P_1 and P_2 are not parallel, so they will intersect (and the distance is therefore zero).

21. Find the error in the following: Let $\mathbf{u}$ be a vector such that $|\mathbf{u}| = 1$. Choose a vector $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 3$ and $|\mathbf{v}| = \sqrt{5}$. Now we have:

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$

Therefore, $\mathbf{u} - \mathbf{v} = \vec{0}$, and so $\mathbf{u} = \mathbf{v}$.

SOLUTION: The setup is impossible. Given that information,

$$\cos(\theta) = \frac{3}{\sqrt{5}} \approx 1.3416$$

And there is no such θ, since the range of the cosine is between ± 1.