Selected Solutions 13.3

13.3, 3. This one had some tricky algebra:

$$\vec{r}'(t) = \langle \sqrt{2}, \mathrm{e}^t, -\mathrm{e}^{-t} \rangle$$

so that the magnitude is:

$$|\vec{r}'(t)| = \sqrt{2 + \mathrm{e}^{2t} + \mathrm{e}^{-2t}} \sqrt{(\mathrm{e}^t + \mathrm{e}^{-t})^2} = \mathrm{e}^t + \mathrm{e}^{-t}$$

Therefore, the arc length is:

$$\int_0^1 |\vec{r}'(t)|\, dt = \Big(\mathrm{e}^t + \mathrm{e}^{-t}\Big|_{t=0}^{1} = \mathrm{e} - \mathrm{e}^{-1}$$

13.3, 11. First, find the curve of intersection between $x^2 = 2y$ and $3z = xy$.

Notice that $y = \frac{1}{2}x^2$ and $z = \frac{1}{3}xy$. Therefore, it might be natural to let x be the free parameter, t and using these equations, the curve of intersection is:

$$x = t \qquad y = \frac{1}{2}t^2 \qquad z = \frac{1}{6}t^3$$

Now find the length of the curve from the origin ($t = 0$) to the point $(6, 8, 36)$ which corresponds to $t = 6$.

Compute the arc length:

$$|\vec{r}'(t)| = \sqrt{1 + t^2 + \frac{1}{4}t^4}$$

We need to integrate this, so some simplification is in order- Notice that this is a perfect square:

$$\left(\frac{1}{2}t^2 + 1\right)^2 = \frac{1}{4}t^4 + t^2 + 1$$

Therefore, the arc length integral becomes:

$$\frac{1}{2}\int_0^6 t^2 + 2\, dt = \frac{1}{2}\left(\frac{1}{3}t^3 + 2t\right|_{t=0}^{6} = 6 + 36 = 42$$

13.3, 16. To re-parameterize with respect to arc length, we:

- Find the arc length function, $s(t) = \int_a^t |\vec{r}'(t)|\, dt$. In this case, it looks worse than it is:

  $$\vec{r}'(t) = \left\langle -2(t^2+1)^{-2}(2t), \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2} \right\rangle = \left\langle \frac{-4t}{(t^2+1)^2}, \frac{-2t^2+2}{(t^2+1)^2} \right\rangle$$

  And

  $$|\vec{r}'(t)|^2 = \frac{16t^2}{(t^2+1)^4} + \frac{(-2t^2+2)^2}{(t^2+1)^4} =$$

  $$\frac{16t^2 + 4t^4 - 8t^2 + 4}{(t^2+1)^4} = \frac{4(t^4 + 2t^2 + 1)}{(t^2+1)^4} = \frac{4(t^2+1)^2}{(t^2+1)^4} = \frac{4}{(t^2+1)^2}$$

  Therefore, the arc length function is:

  $$s(t) = \int_0^t \frac{2}{u^2+1}\, du = 2\left(\tan^{-1}(t) - \tan^{-1}(0)\right) = 2\tan^{-1}(t)$$

- Invert to get t in terms of s:

$$s = 2\tan^{-1}(t) \quad \Rightarrow \quad t = \tan\left(\frac{s}{2}\right)$$

- Now we simply substitute back into the expression for the curve $\vec{r}(t)$, noting that the parameterization is with respect to arc length s:

$$\vec{r}(s) = \left\langle \frac{2}{\tan^2(s/2)+1} - 1, \frac{2\tan(s/2)}{\tan^2(s/2)+1} \right\rangle$$

  For fun, you might try to simplify this (not necessary at this point). If you do, you end up with: $\vec{r}(s) = \langle \cos(s), \sin(s) \rangle$

13.3, 17. This is pretty straightforward computation:

$$\vec{r}\,'(t) = \langle 2\cos(t), 5, -2\sin(t) \rangle \quad \Rightarrow \quad |\vec{r}\,'(t)| = \sqrt{4\cos^2(t) + 25 + 4\sin^2(t)} = \sqrt{29}$$

Therefore,

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{\sqrt{29}}\langle 2\cos(t), 5, -2\sin(t) \rangle$$

Now compute the unit normal vector $\mathbf{N}(t)$. Note: For easier computation, keep the constant out in front.

$$\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin(t), 0, -2\cos(t) \rangle \quad \Rightarrow \quad |\mathbf{T}'(t)| = \frac{2}{\sqrt{29}}$$

Therefore,

$$\mathbf{N}(t) = \frac{1}{2}\langle -2\sin(t), 0, -2\cos(t) \rangle$$

13.3, 19. The algebra trick from Exercise 3 shows up again here:

$$\mathbf{r}'(t) = \langle \sqrt{2}, \mathrm{e}^t, -\mathrm{e}^{-t} \rangle$$

And the magnitude:

$$|\mathbf{r}'(t)| = \sqrt{2 + \mathrm{e}^{2t} + \mathrm{e}^{-2t}} = \sqrt{(\mathrm{e}^t + \mathrm{e}^{-t})^2} = \mathrm{e}^t + \mathrm{e}^{-t}$$

so that the unit tangent vector is:

$$\mathbf{T}(t) = \frac{1}{\mathrm{e}^t + \mathrm{e}^{-t}}\langle \sqrt{2}, \mathrm{e}^t, -\mathrm{e}^{-t} \rangle$$

There is a compound fraction there that should probably be simplified:

$$\frac{1}{\mathrm{e}^t + \mathrm{e}^{-t}} = \frac{\mathrm{e}^t}{\mathrm{e}^{2t} + 1}$$

Multiplying through, we have:

$$\mathbf{T}(t) = \frac{1}{\mathrm{e}^{2t} + 1}\left\langle \sqrt{2}\,\mathrm{e}^t, \mathrm{e}^{2t}, -1 \right\rangle$$

Recall that the derivative may be computed as:

$$(f(t)\mathbf{u}(t))' = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

or in this case,

$$\mathbf{T}'(t) = -\frac{2\mathrm{e}^{2t}}{(\mathrm{e}^{2t}+1)^2}\langle\sqrt{2}\,\mathrm{e}^t, \mathrm{e}^{2t}, -1\rangle + \frac{1}{\mathrm{e}^{2t}+1}\left\langle\sqrt{2}\,\mathrm{e}^t, 2\mathrm{e}^{2t}, 0\right\rangle$$

Some algebra later...

$$\mathbf{T}'(t) = \frac{1}{(\mathrm{e}^{2t}+1)^2}\left\langle\sqrt{2}\,\mathrm{e}^t(1-\mathrm{e}^{2t}), 2\mathrm{e}^{2t}, 2\mathrm{e}^{2t}\right\rangle$$

And a LOT of algebra later, we can compute $\mathbf{N}(t)$ as:

$$\frac{1}{\mathrm{e}^{2t}+1}\langle 1-\mathrm{e}^{2t}, \sqrt{2}\,\mathrm{e}^t, \sqrt{2}\,\mathrm{e}^t\rangle$$

13.3, 43. There are some shortcuts we can make here. First, some computations and evaluations:

$$\mathbf{r}'(t) = \langle 2t, 2t^2, 1\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{4t^4+4t^2+1} = 2t^2+1$$

At $t = 0$, we have the following:

$$\mathbf{r}'(1) = \langle 2, 2, 1\rangle \quad \Rightarrow \quad |\mathbf{r}'(1)| = 3$$

Therefore,

$$\mathbf{T}(1) = \frac{1}{3}\langle 2, 2, 1\rangle$$

To compute N, we need $\mathbf{T}$ in terms of t:

$$\mathbf{T}(t) = \frac{1}{2t^2+1}\langle 2t, 2t^2, 1\rangle$$

Use the product rule to differentiate:

$$\mathbf{T}'(t) = \frac{-4t}{(2t^2+1)^2}\langle 2t, 2t^2, 1\rangle + \frac{1}{2t^2+1}\langle 2, 4t, 0\rangle$$

Now, we only need $\mathbf{N}(1)$, so evaluate $\mathbf{T}'(1)$, then make it a unit vector.

$$\mathbf{T}'(1) = \frac{1}{9}\langle -2, 4, -4\rangle = \frac{2}{9}\langle -1, 2, -2\rangle$$

To find $\mathbf{N}(1)$ a unit vector, we can really just ignore the 2/9 and just make the vector have magnitude 1:

$$\mathbf{N}(1) = \frac{1}{3}\langle -1, 2, -2\rangle$$

Now, the binormal vector is the cross product of $\mathbf{T}$ and $\mathbf{N}$:

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{3}\langle -2, 1, 2\rangle$$

13.3, 44. The computations are very similar to the previous problem:

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), -\tan(t)\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{1+\tan^2(t)} = \sec(t)$$

Note that secant is positive close to $t = 0$. Now, compute these at $t = 0$ for $\mathbf{T}(0)$:

$$\mathbf{T}(0) = \langle 0, 1, 0\rangle$$

Simplifying before we differentiate $\mathbf{T}$, we have:

$$\mathbf{T}(t) = \langle -\sin(t)\cos(t), \cos^2(t), -\sin(t)\rangle$$

so that differentiating,

$$\mathbf{T}'(t) = \langle \sin^2(t) - \cos^2(t), -2\sin(t)\cos(t), -\cos(t)\rangle$$

Now we can compute $\mathbf{N}(0)$:

$$\mathbf{N}(0) = \frac{1}{\sqrt{2}}\langle -1, 0, -1\rangle$$

Finally, compute the binormal vector using the cross product:

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle$$

13.3, 45. Compute the equations of the normal plane (whose normal vector is $\mathbf{T}$) and the osculating plane (whose normal vector is $\mathbf{B}$), for the given function at $t = \pi$. First compute the TNB vectors as before:

$$\mathbf{r}'(t) = \langle 6\cos(3t), 1, -6\sin(3t)\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{36\cos^2(3t) + 1 + 36\sin^2(t)} = \sqrt{37}$$

Therefore,

$$\mathbf{T}(\pi) = \frac{1}{\sqrt{37}}\langle -6, 1, 0\rangle$$

And for $\mathbf{N}$, we need to differentiate $\mathbf{T}$:

$$\mathbf{T}'(t) = \frac{1}{\sqrt{37}}\langle -18\sin(3t), 0, -18\cos(3t)\rangle$$

and we see that (again, ignore the constant and just make $\mathbf{N}$ a unit vector):

$$\mathbf{N}(\pi) = \langle 0, 0, 1\rangle$$

Finally, the binormal vector is the cross product:

$$\mathbf{B}(\pi) = \frac{1}{\sqrt{37}}\langle 1, 6, 0\rangle$$

The planes will intersect the curve at $t = \pi$, or at the point $(0, \pi, -2)$, so now we have the normal vectors and the point:

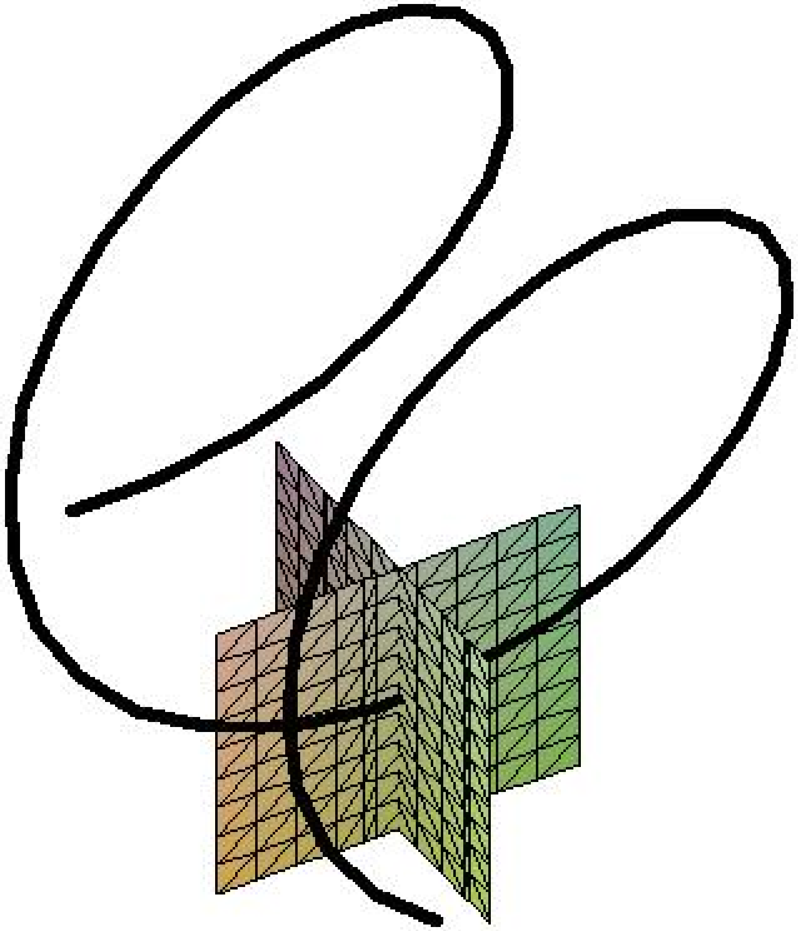

Figure 1: Figure showing the curve, the normal plane (cuts the curve) and the osculating plane (embeds the curve) for Exercise 45, section 13.3.

- The normal plane (we use a nice scalar multiple of $\mathbf{T}(\pi)$

$$-6(x-0)+1(y-\pi)+0(z+2)=0 \quad \Rightarrow \quad -6x+y-\pi=0$$

- The osculating plane uses $\mathbf{B}$ as the normal:

$$(x-0)+6(y-\pi)+0(z+2)=0 \quad \Rightarrow \quad x+6y-6\pi=0$$

13.3, 46. Same type of computations as 45, but the length is a bit messy:

$$\mathbf{r}'(t)=\langle 1,2t,3t^2\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)|=\sqrt{1+4t^2+9t^4}$$

Therefore,

$$\mathbf{T}(1)=\frac{1}{\sqrt{14}}\langle 1,2,3\rangle$$

Notice that we could stop here to get the equation for the normal plane using the vector $\langle 1,2,3\rangle$:

$$1(x-1)+2(y-1)+3(z-1)=0 \quad \Rightarrow \quad x+2y+3z-6=0$$

For $\mathbf{N}$, we need to differentiate $\mathbf{T}$ which is messy. If done correctly, you should get

$$\mathbf{N}(1) = \frac{-2}{14^{3/2}}\langle 11, 8, -9\rangle$$

Finally, the binormal vector is the cross product. If you consider the problem, we don't need to carry along the constants as long as we remember to make the vector a unit vector at the end. In this case,

$$\langle 1, 2, 3\rangle \times \langle 11, 8, -9\rangle = \langle -42, 42, -14\rangle$$

Factor 14 out and make it a unit vector for $\mathbf{N}$ (or, if we don't need $\mathbf{N}$, but just the osculating plane, just factor out 14):

$$\mathbf{N}(1) = \frac{1}{\sqrt{19}}\langle -3, 3, -1\rangle$$

and the osculating plane is:

$$-3(x-1) + 3(y-1) - (z-1) = 0 \quad \Rightarrow \quad -3x + 3y + z - 1 = 0$$

13.3, 49. The normal plane is perpendicular to the tangent vector (either $\mathbf{r}'(t)$ or $\mathbf{T}(t)$):

$$\mathbf{r}'(t) = \langle 3t^2, 3, 4t^3\rangle$$

We need to find a t so that $\mathbf{r}'(t)$ is parallel to $\langle 6, 6, -8\rangle$ or factoring 2 out, we need $\mathbf{r}'(t)$ to be parallel to $\langle 3, 3, -4\rangle$.

Now we see that they are parallel by taking $t = -1$, which corresponds to the point $(-1, -3, 1)$.

13.3, 59. A fun problem- First, from the information given, you should get (in angstroms):

$$\vec{r}(t) = \langle 10\cos(t), 10\sin(t), 34t/(2\pi)\rangle$$

Then the arc length for one turn:

$$\int_0^{2\pi} \sqrt{100 + \frac{34}{2\pi}^2}\, dt \approx 71.441177$$

in angstroms. Multiply that by 2.09×10^8 to get the full length- about 2.07 meters!

Section 13.4

13.4, 15. Antidifferentiate to find the velocity and position vectors, remembering to use the arbitrary constants!

$$\mathbf{a}(t) = \langle 1, 2, 0\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \langle t + c_1, 2t + c_2, c_3\rangle$$

With the initial velocity $\langle 0, 0, 1\rangle$, these are the constants:

$$\mathbf{v}(t) = \langle t, 2t, 1\rangle \quad \Rightarrow \quad \mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + c_1, t^2 + c_2, t + c_3\right\rangle$$

With the initial position $\langle 1, 0, 0\rangle$, we get

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t^2 + 1, t^2, t\right\rangle$$

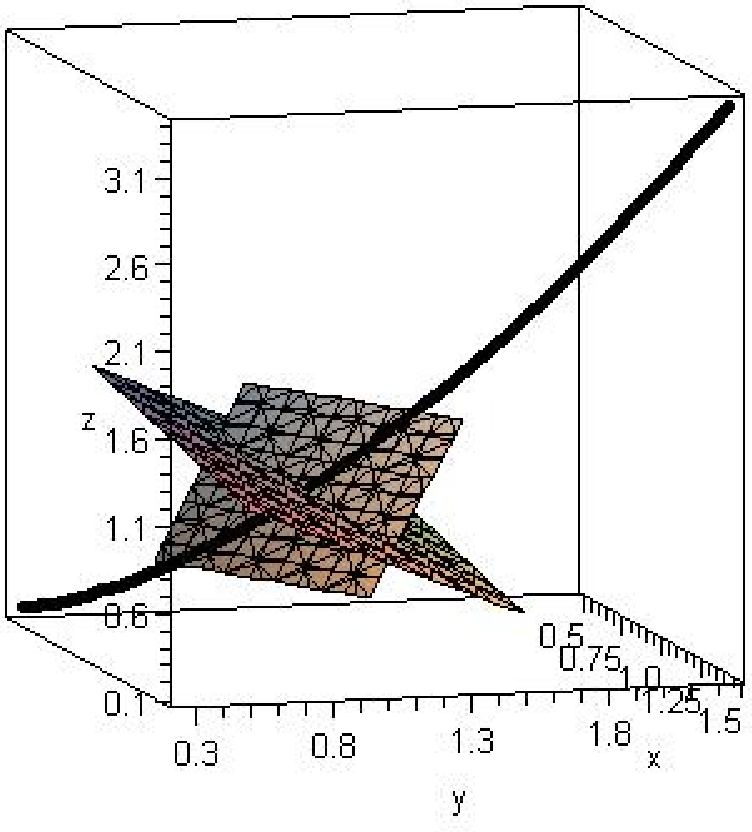

Figure 2: Figure showing the curve, the normal plane (cuts the curve) and the osculating plane (embeds the curve) for Exercise 46, section 13.3.

13.4, 19. With $\mathbf{r}(t)$ given, the speed is the magnitude of velocity. We can find the minimum of that by finding the minimum of the square of the magnitude,

$$|\mathbf{r}'(t)|^2 = 4t^2 + 25 + (2t - 16)^2 = 8t^2 - 64t + 281$$

To find the minimum, take the derivative and set to zero:

$$16t - 64 = 0 \quad \Rightarrow \quad t = 4$$

Does a minimum occur at $t = 4$ or does a maximum? The second derivative is 16 (positive), so at $t = 4$ we have a minimum- The minimum speed is found by substituting into the formula for the (square of) the speed:

$$\sqrt{153} = 3\sqrt{17}$$

13.4, 21. Information we are given in the problem: Since the force is directed upwards with a magnitude of 20, the force vector is:

$$\vec{F}(t) = 20\vec{k}$$

Force is also mass times acceleration, so

$$20\vec{k} = 4\,a(\vec{t}) \quad \Rightarrow \quad \vec{a}(t) = 5\vec{k}$$

Now we integrate to find the velocity:

$$v(t) = \int \vec{a}(t)\,dt = \langle c_1, c_2, 5t + c_3\rangle$$

And $\vec{v}(0) = \langle 1, -1, 0\rangle$, so these are our constants, and

$$\vec{v}(t) = \langle 1, -1, 5t\rangle$$

Position is found by integrating velocity,

$$\vec{r}(t) = \left\langle t + c_1, -t + c_2, \frac{5}{2}t^2 + c_3 \right\rangle$$

The object begins at the origin, so $\vec{r}(0) = \vec{0}$, therefore the constants are zero and our position function is:

$$< t, t, (5/2)t^2\rangle$$

13.4, 23. A projectile is fired with an initial speed of 500 meters per second and an angle of elevation of 30 degrees.

Before continuing, use the results of Example 5 (Equation 4), on p. 841 to write:

$$x(t) = 500\cos(30)t = 250\sqrt{3}\,t$$

And for the $y-$coordinate,

$$y(t) = 500\sin(30)\,t - \frac{1}{2}gt^2 = 250t - \frac{9.8}{2}t^2 = 250t - 4.9t^2$$

Our position vector is therefore $\vec{r}(t) = \langle 250\sqrt{3}\,t, 250t - 4.9t^2\rangle$

To find the range of the projectile, we determine the times at which the height was zero. These should correspond to the time at launch and the time at landing:

$$250t - 4.9t^2 = 0 \quad \Rightarrow \quad t(250 - 4.9t) = 0 \quad \Rightarrow \quad t = 0, t \approx 51.02$$

At 51.02 seconds, the $x-$coordinate was approximately $22,092$ meters, or about 22 km.

To find the speed at impact, we substitute $t = 51.02$ into the expression for $|\vec{v}(t)|$:

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \sqrt{250^2 \cdot 3 + (250 - 9.8t)^2}$$

At $t \approx 51.02$, we should find that the speed was approximately 500 meters per second.

13.4, 25. Similarly (to Exercise 23 and Example 5), if a ball is thrown at an angle of 45 degrees to the ground, we can write:

$$x(t) = \frac{v_0}{\sqrt{2}}t \qquad y(t) = \frac{v_0}{\sqrt{2}}$$