

Ants on a Doughnut

Adopted from Neal Brand, University of North Texas

Two ants are exploring a doughnut and notice that there are two regions on the doughnut which have particularly good supplies of sugar and grease. The two regions are as far apart on the doughnut as they can be. The ants must leave a path for all the other ants to follow. In order to be efficient the ants wish to find the shortest path between the good regions.

The ants notice that the doughnut is obtained by rotating the circle

$$(x - 2)^2 + z^2 = 1$$

(in the xz -plane) about the z -axis. This puts the doughnut in xyz -space. The areas rich in sugar and grease are at the positions $(3, 0, 0)$ and $(-3, 0, 0)$.

The first ant, Grant, suggests that the shortest route would be to follow the path of $(3, 0, 0)$ as it rotates about the z -axis as in Figure 1 Antonio, the second ant suggests that the “inside”

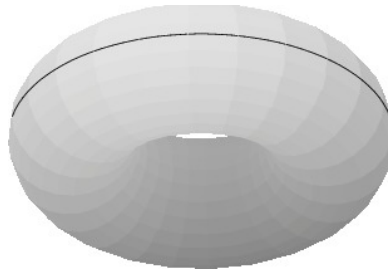


Figure 1: Staying on the outside of the doughnut.

circle is smaller, so it would make sense to follow the circle $(x - 2)^2 + z^2 = 1$ half way around so they are at the “small” circle, follow it half way around to the point $(-1, 0, 0)$, and then go around the circle from $(-1, 0, 0)$ to $(-3, 0, 0)$. (See Figure 2)

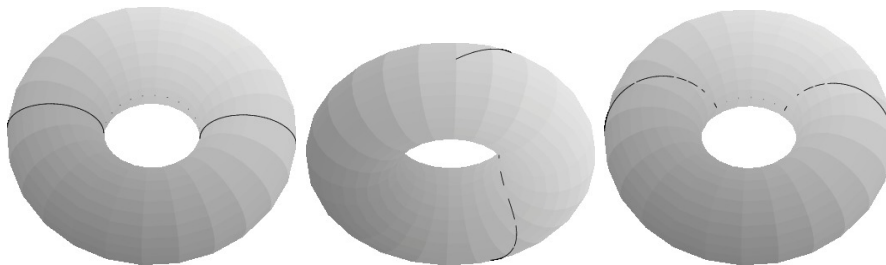


Figure 2: (LEFT) Moving to the inside immediately, around the inside and back to the outside. (CENTER) Moving gradually around in both circles at the same time. (RIGHT) A little of the two previous trials.

After hearing Antonio’s idea Grant thinks that it would be better not to go directly around the circle $(x - 2)^2 + z^2 = 1$ before starting around the z -axis rotation. Grant says “Why not go around both circles at the same time? Perhaps we could go around both circles simultaneously and save some distance.” (See Figure 2.)

Antonio then suggests going around both at the same time part way until they hit the ‘inside circle’ on the xy -plane, around the inside circle for a ways and finally around both until they arrive at $(-3, 0, 0)$. Their path could be symmetric about the yz -plane. (See Figure 2.) Which route is shortest among those discussed?

Goal of the lab: Help the ants find a path that is shorter than any of those currently given. In particular, try to do this using a family of functions that you can optimize. (Something to think about: Is there a shortest path?)

Here are some steps to get you started:

1. We need parametric equations for the surface of the doughnut (torus). In your write up, you may assume that the equations are what we present in class (you do not have to explain them).
2. To organize your paths, plot them in the (β, α) plane. What are the beginning and ending points?
3. Try some new paths to get an idea of what you need to do to get the shortest path. We’ll look at a couple of Maple worksheets to help.

Maple Commands for the Surface

```
restart;
with(plots):
f:=cos(beta)*(cos(alpha)+2);
g:=sin(beta)*(cos(alpha)+2);
h:=sin(alpha);
plot3d([f,g,h],alpha=0..2*Pi,beta=0..2*Pi,scaling=constrained,
style=patchnogrid,shading=ZHUE);
```

Maple Commands for a Path

These commands assume the first section has been done. We will find a parametric representation for the path- That means α, β are functions of time. Then these are substituted into the equations for the torus.

NOTE: The end of the line for each of these is for readability. Each of these would be typed on its own line.

For this example, $\alpha = 0$ and $\beta = \pi t$ so that, as t goes from 0 to 1, (α, β) goes from $(0, 0)$ to $(0, \pi)$. This path on the torus goes from $(3, 0, 0)$ to $(-3, 0, 0)$.

```
Path1Functions:=subs({alpha=0, beta=t*Pi},f), subs({alpha=0, beta=t*Pi},g),
subs({alpha=0, beta=t*Pi},h);
```

```
Path1:=spacecurve([Path1Functions, t=0..1],color=black,thickness=5):
```

```
Torus1:=plot3d([f,g,h],alpha=0..2*Pi,beta=0..2*Pi,scaling=constrained,
```

```
style=patchnogrid,shading=ZHUE):
```

```
display3d({Torus1, Path1});
```

The Arc Length

If we assume that α, β are functions of time, $0 \leq t \leq 1$, and these have been substituted into the equations for x, y and z for the surface of the torus, then we can think of x, y and z as functions of time. The arc length of this path in 3-d is:

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

so, continuing our commands in Maple, we could write:

```
dx:=diff(Path1Functions[1],t);
```

```
dy:=diff(Path1Functions[2],t);
```

```
dz:=diff(Path1Functions[3],t);
```

```
PathLength:=Int(sqrt(dx^2+dy^2+dz^2),t=0..1);
```

```
evalf(PathLength);
```