Ants on a Doughnut[1]

Two ants are exploring a doughnut and notice that there are two regions on the doughnut which have particularly good supplies of sugar and grease. The two regions are as far apart on the doughnut as they can be. The ants must leave a path for all the other ants to follow. In order to be efficient the ants wish to find the shortest path between the good regions.

The ants notice that the doughnut is obtained by rotating the circle $(x-2)^2+z^2=1$ (in the xz-plane) about the z-axis. This puts the doughnut in xyz-space. The areas rich in sugar and grease are at the positions $(3,0,0)$ and $(-3,0,0)$.

The first ant, Grant, suggests that the shortest route would be to follow the path of $(3,0,0)$ as it rotates about the z-axis as in Figure 1 Antonio, the second ant suggests

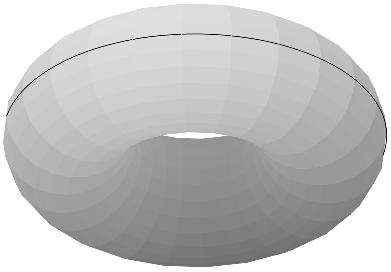

Figure 1: Staying on the outside of the dougnut.

that the "inside" circle is smaller, so it would make sense to follow the circle $(x-2)^2+z^2=1$ half way around so they are at the "small" circle, follow it half way around to the point $(-1,0,0)$, and then go around the circle from $(-1,0,0)$ to $(-3,0,0)$. (See Figure 2)

After hearing Antonio's idea Grant thinks that it would be better not to go directly around the circle $(x-2)^2+z^2=1$ before starting around the z-axis rotation. Grant says "Why not go around both circles at the same time? Perhaps we could go around both circles simultaneously and save some distance." (See Figure 3.)

Antonio then suggests going around both at the same time part way until they hit the 'inside circle' on the xy-plane, around the inside circle for a ways and finally around both until they arrive at $(-3,0,0)$. Their path could be symmetric about the yz-plane. (See Figure 4.)

Which route is shortest among those discussed?

Follow the steps below to help the ants find a short path between the two regions.

1. Think of each point on the torus (or doughnut) as being specified by two angles, say α and β. One angle should measure how far the circle $(x-2)^2+z^2=1$

[1] `http://www.math.unt.edu/~brand/projects/calc3/calc3.htm`

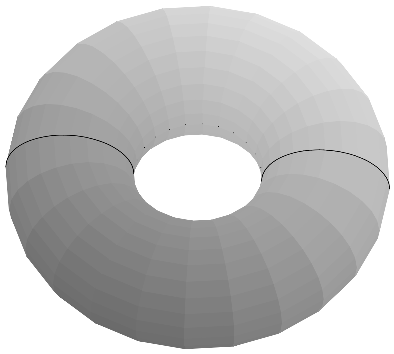

Figure 2: Moving to the inside immediately, around the inside and back to the outside.

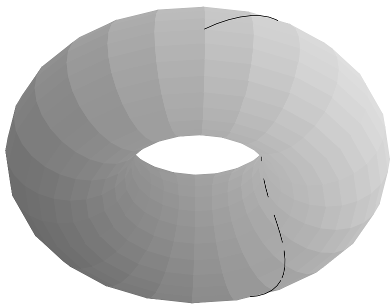

Figure 3: Moving gradually around in both the α and β parameters.

has revolved around the z-axes while the other should measure how far around the circle $(x-2)^2+z^2=1$ you rotate. Find formulas which give x, y, and z in terms of the two angles.

2. Make a graph of the paths originally suggested by the two ants in the $\alpha\beta$-plane. Find the length of Grant's original path and Antonio's original path. (Remember, you must find the length in three space, not in the $\alpha\beta$-plane.) Which is shorter? Would your answer change if the circle being rotated about the z-axis were moved closer to or further from the z-axis?

3. Now make a graph in the $\alpha\beta$-plane of the second path suggested by Grant. Find the length of this path. You may have trouble computing the integral giving the exact path length. If so, you may use Maple, or any other computer program (including one you write yourself) to estimate the integral which gives the path length you seek.

4. Now make a graph in the $\alpha\beta$-plane of the second path suggested by Antonio. Find the optimal point on the "inside" circle where the ants should aim before starting around the inside circle. (This will require some first semester calculus. You are allowed to use the fact that $\partial_x \int_a^b f(x,t)dt = \int_a^b \partial_x f(x,t)dt$.) Assum-

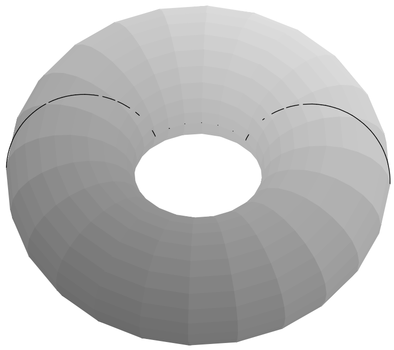

Figure 4: In the first third of the path, moving gradually around in both the α and β parameters to a point on the inner circle, in the second third of the path moving around the inner circle and for the third third of the the path taking a path symmetric to that of the first third back to the outside of the dougnut.

ing they aim for this point, what is the distance around the doughnut? (You may wish to use the computer to approximate the optimal point and to compute the distance around.)

5. Of the four paths suggested which is best? Do you think there is a path shorter than any of the four you studied?