Lab 5: Multivariate Calculus

Your Names Here

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Since we are running a bit short of time for this lab, this lab will consist of answering several questions from Calculus III. Unlike previous labs, in this lab, you may simply write a short answer for each question in LaTeX. You should use figures where appropriate.
NOTE: Because of the 3-d graphics, please be sure you are running Maple 12 and not Maple 13.
NOTE: You may use this document as a starting point, but remove everything you don't need.

Maple Commands, part I:

In this lab you will need to use the Maple commands for plotting graphs, level curves and gradient vector fields. To load these commands, begin by typing:

```
with(plots):
```

Recall that the command for plotting surfaces is `plot3d`:

```
plot3d(x^2+y^2,x=-2..2,y=-2..2);
```

The contourplot command is used to plot level curves:

```
contourplot(x^2+y^2,x=-2..2,y=-2..2,contours=30,scaling=constrained);
```

The option `contours=30` tells Maple to plot 30 level curves. Finally, the `gradplot` command is used to plot the gradient vector field $\nabla f(x, y)$ of a function $z = f(x, y)$.

```
gradplot(x^2+y^2,x=-2..2,y=-2..2,scaling=constrained,arrows=slim);
```

Use the display command to view the level curves and gradient vector field on the same plot.

```
plot1:=contourplot(x^2+y^2,x=-2..2,y=-2..2,contours=30,scaling=constrained):
plot2:=gradplot(x^2+y^2,x=-2..2,y=-2..2,scaling=constrained,arrows=slim):
display(plot1,plot2);
```

Notice that the gradient vectors are perpendicular to the level curves, as expected.

Lab Questions, Part I

1. Consider the function $f(x, y) = x^3 - x^2 - 3x + 8y - y^3 + xy$ on the domain $[-3, 3] \times [-3, 3]$.
   - Plot its graph, level curves, and gradient vector field on this domain. Plot the level curves and gradient vector field together. Discuss how we might find and classify the critical points of f using this graph.
   - Algebraically find and classify all of the critical points of f using the second derivative test, if it can be applied.
2. Same as Question 1, but using $g(x, y) = y^2 + 3x - x^3$.

Maple Commands, Part II

Double integrals are computed by iterating a single integral. For example (white space included for readability only):

$$\int_0^3 \int_x^4 xy^2 \, dy \, dx \quad \text{is computed as} \quad \texttt{int(int(x*y^2, y=x..4), x=0..3);}$$

Parametric curves in three dimensions are plotted using the `spacecurve` command. For example,

```
x:=2*sin(t);   y:=3*cos(t);  z:=1-sin(3*t);
spacecurve([x,y,z],t=0..15)
```

Surfaces can also be parameterized. They are mappings of the plane into three dimensions. Examples and the plots:

- If $z = f(x, y)$, we can write $x = u$, $y = v$, $z = f(u, v)$ to get three functions of (u, v). For example, if $z = x^2 - y^2$, then we could parameterize the surface and plot it as:

  ```
  restart; with(plots):  #Clear everything out
  x:=u; y:=v; z:=u^2-v^2;
  plot3d([x,y,z],u=-1..2,v=-1..2);
  ```

- The following three equations will plot the surface of a sphere. In this case, if we have a point on the surface of a sphere, u is the angle that point makes in the $xy-$plane, and v is the angle the point makes with the (positive) $z-$axis:

  ```
  x=sin(v)*cos(u); y:=sin(v)*sin(u); z:=cos(v);
  plot3d([x,y,z], u=0..2*Pi,v=0..Pi);
  ```

  Compare this to the plot of the sphere. See if you can reason out why this is so bad.

  ```
  x:='x'; y:='y';
  f:=sqrt(1-x^2-y^2);
  plot3d({f,-f},x=-2..2,y=-2..2, scaling=constrained);
  ```

- A surface of revolution from Calc II can be parameterized in three dimensions. If $y = f(x)$ and we rotate about the $x-$axis, then let u be the parameter measuring where we are on the $x-$axis, and v is the angle of rotation. The plane of rotation is parallel to the yz plane, and

  $$x = u \qquad y = f(u)\cos(v) \qquad z = f(u)\sin(v)$$

  For example, to rotate the curve

  ```
  x:='x'; y:='y'; z:='z';
  f:=2+sin(u)-cos(u); x:=u; y:=f*cos(v); z:=f*sin(v);
  plot3d([x,y,z],x=0..11,v=0..2*Pi);
  ```

  To put a curve on the surface of your object, we recall that it is a mapping from the real line into 3-d. Therefore, if u and v are both functions of time, then the plot above would be a curve and not a surface:

  ```
  x:='x'; y:='y'; z:='z'; u:='u'; v:='v';
  f:=2+sin(u)-cos(u); x:=u; y:=f*cos(v); z:=f*sin(v);
  A:=plot3d([x,y,z],u=0..11,v=0..2*Pi);
  u:=11*sin(t); v:=2*Pi*t;
  B:=spacecurve([x,y,z],t=0..1,color=black, thickness=2);
  display3d({A,B});
  ```

Lab Questions, continued

3. Look up the definition of the center of mass of a lamina that occupies a region D with density function $\rho(x, y)$. Find the center of mass if

$$D = \{(x, y) \,|\, 0 \le y \le \sin(x),\ 0 \le x \le \pi\}, \quad \rho(x, y) = xy$$

4. Given the surface represented by:

$$x = 2\cos(\theta) + r\cos(\theta/2) \quad y = 2\sin(\theta) + r\cos(\theta/2) \quad z = r\sin(\theta/2)$$

where $-1/2 \le r \le 1/2$, and $0 \le \theta \le 2\pi$, graph it, then put an interesting curve on its surface and plot them together (be creative!).

5. We're told that the surface area given by a surface parameterized by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

for $(u, v) \in D$ is given by:

$$A(s) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Plot the surface defined by:

$$x = (1-u)(3 + cos(v))cos(4\pi u) \quad y = (1-u)(3 + cos(v))sin(4\pi u) \quad z = 3u + (1-u)\sin(v)$$

where $0 \le u \le 1$ and $0 \le v \le 2\pi$. Now compute a numerical approximation for the double integral representing the surface area.