Newton's Method in 2 Dimensions[1]

The ordinary Newton's Method uses the linear approximation to find an approximate solution to an equation of the form $f(x) = 0$. Basically, if x_0 is an initial approximation to the solution, then the tangent line to $y = f(x)$ at $x = x_0$ intersects the $x-$axis at a point $(x_1, 0)$, and x_1 is usually a better approximation to the solution than x_0. So the process can be iterated using x_1, and a short derivation shows that at each stage,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This may be automated in Maple by defining the function

```
> newt:= x -> evalf(x-f(x)/Df(x));
```

Here is a full example. We will plot the curve to estimate a starting value x_0- Here, after viewing the plot, we decide on $x_0 = 0.8$. (Note about typing the for loop: Use the shift then enter keys to get a new line without Maple evaluating the expression).

```
> Digits:=16;
> f:=x-> cos(x)-x;
> Df:=D(f);
> newt:= x -> evalf(x-f(x)/Df(x));
> plot(f(x),x=-Pi..Pi);
> t:=1.2;
> for i from 1 to 4 do
    y:=newt(t);
    if abs(y-t)<10^(-8)
    then printf("Done on iterate %d", i);
         printf(" and the solution is %f\n", y);
         break;
    else
         t:=y;
    end
   od;
```

NOTES: New in Maple- The *D* command, and look over the "for loop", which is ended by the `od` line. There is also an `if` statement.

[1]Adopted from *CalcLabs with Maple for Stewart's Multivariate Calculus*, P.B. Yasskin and A. Belmonte

Two dimensional Newton's Method

The two dimensional Newton's Method works in the same way- We are looking for an ordered pair (x, y) that solves the pair of equations

$$f(x, y) = 0 \qquad g(x, y) = 0$$

If (x_0, y_0) is an initial approximation to the solution, then the tangent plane to $z = f(x, y)$ at (x_0, y_0) and the tangent plane to $z = g(x, y)$ at (x_0, y_0) intersect the $xy-$plane at a common point, $(x_1, y_1, 0)$. Hopefully (x_1, y_1) is a better approximation to the solution than (x_0, y_0).

Lab Questions

1. Derive equations for x_{i+1} and y_{i+1} like we had for the original Newton's Method. Your solutions will depend on f, f_x, f_y, g, g_x, g_y, all evaluated at (x_i, y_i). Your answer should be in the form:

   $$x_{i+1} = x_i - \underline{\qquad\qquad} \qquad\qquad y_{i+1} = y_i - \underline{\qquad\qquad}$$

2. Construct a single Maple function, `newt2d` which acts on an initial approximation and produces the next approximation. Here is a simple example of a function that takes in two numbers and produces two numbers for iteration:

```
> newtex:=(x,y)->(3*x-5*y, 2*x+y);
> g:=newtex(1,2);
> newtex(g);
```

3. Use your Maple function to find all solutions to each of the following pairs of equations. You will need to plot the two equations using `implicitplot` to get an initial approximation to each solution. Iterate enough so that the maximum difference between two successive iterations is no more than 10^{-10}. You can use `fsolve` to check your solutions.

   (a) $x + y - \cos(x) + \sin(y - 1) = 0$ and $x^4 + y^4 - 2xy = 0$

   (b) $5x - 3y = -2$ and $2x - 2y = -3$

   (c) $y^3x - x^3y + x^2y^2 = 7$ and $2x^4 + 3y^4 = 74$