FIRST ORDER AUTONOMOUS DIFFERENTIAL EQUATIONS

1. Introduction

A first order, autonomous differential equation is of the form:

$$y' = f(y)$$

(1) A direction field is a plot of local slopes where the axes are t and y. A *phase plot* is a plot of y' versus y.
(2) An *equilibrium solution* is a constant solution, $y(t) = c$ so that $f(c) = 0$. These solutions are also called *fixed points* or *critical points*.
(3) An equilibrium solution is said to be an *attracting* solution if there is an interval of initial values about $y = c$ so that all solutions tend to c as $t \to \infty$.
(4) An equilibrium solution is said to be a *repelling* solution if there is an interval about $y = c$ so that all solutions tend away from $y = c$.

2. Goal of the Lab

The goal of this lab is to explore first order autonomous differential equations, learn some vocabulary associated with them, and to do graphical analysis of the solutions.

3. Lab Questions

(1) From your last lab, you discussed the direction field for DEs of the form $y' = f(y)$. Use your conclusions to discuss the following observation:

> "No solution to $y' = f(y)$ can oscillate. All solutions will either be constant, monotone increasing, or monotone decreasing."

(2) The relationship of the phase plot to the direction field: Suppose we are given the differential equation:

$$y' = f(y) = \frac{1}{10}y(3 - y)(y + 1)^2$$

(a) Graph $f(y)$ versus y and compare this to the direction field. Is it possible to predict the long term behavior of a particular solution by its position in the phase plot?
(b) What are the equilibrium solutions? Classify each as *attracting, repelling,* or *neither.* For each equilibrium, was it possible to predict this based only on the phase plot? Hint: Consider $\frac{df}{dy}$ at each equilibrium.
(c) Comment on the following observation for autonomous, first order equations:

> "The behavior of all solutions to $y' = f(y)$ is organized around the equilibrium solutions"

(3) Consider the family of differential equations of the form:

$$y' = ky - y^3$$

where k is a constant.

(a) Solve for the equilibrium solutions in terms of k (you can do this by hand).
(b) Perform a phase plot for different values of k, and state in words the effect that k has on the graph of $ky - y^3$. Consider the following situations:
  - If $k < 0$, how many equilibria are there, and what type are they?
  - If $k = 0$, answer the same question.
  - If $k > 0$, answer the same question.

Side Remark: The phenomena you are talking about is called a **bifurcation**- *that is, changing a parameter results in the creation/destruction of equilibrium solutions! In fact, this particular situation is called a* **pitchfork bifurcation**.

(4) Some general questions:

(a) Is it possible to have two attracting equilibria with no repelling equilibrium in between? To answer this, consider the phase plot of $f(y)$ versus y.
(b) Construct your own autonomous, first order differential equation modeling population, that would have the correspond to the following behavior:
"If the population falls below 2, the population dies off. If the population is above 2 and less than 20, the population will tend to 20. If the population is greater than 20, there is not enough food so the population will tend back towards 20."
Verify the behavior by providing a phase plot and direction field.