Notes about 32 and 33, Section 4.1

To prove that some set is a subspace, we have to:

- Show that $\vec{0}$ is in the set.
- Show that, if $\vec{x}$ is in the set, and $\vec{y}$ is in the set, then $\vec{x} + \vec{y}$ must be in the set.
- Show that, if $\vec{x}$ is in the set, and c is any scalar, then $c\mathbf{x}$ is in the set.

In particular, we have to know how to show that a vector is an element of our set. Here's a step by step solution to Exercise 33.

33. Given subspaces H and K of vector space V, we define the sum:

$$H + K = \{\mathbf{u} + \mathbf{v} \,|\, \mathbf{u} \in H \text{ and } \mathbf{v} \in K\}$$

Therefore, to show that something is in $H + K$, we have to be able to write it in terms of one vector in H plus one vector in K. Now we show that $H + K$ is a subspace.

- Is $\vec{0} \in H + K$. Since H, K are subspaces, then $\vec{0} \in H$ and $\vec{0} \in K$. We can write:

  $$\vec{0} = \vec{0} + \vec{0}$$

  where the first zero is from H and the second is from K. Therefore, $\vec{0}$ is in $H + K$.

- Show that $H + K$ is closed under addition.

  SOLUTION: Let $\mathbf{x}$ and $\mathbf{y}$ each be in $H + K$. Therefore, we can write:

  $$\begin{array}{rlll} \mathbf{x} & = \mathbf{x}_H & + \mathbf{x}_K & \text{for some } \mathbf{x}_H \in H, \mathbf{x}_K \in K \\ \mathbf{y} & = \mathbf{y}_H & + \mathbf{y}_K & \text{for some } \mathbf{y}_H \in H, \mathbf{y}_K \in K \\ \hline \mathbf{x} + \mathbf{y} & = (\mathbf{x}_H + \mathbf{x}_K) & + (\mathbf{y}_H + \mathbf{y}_K) = (\mathbf{x}_H + \mathbf{y}_H) + (\mathbf{x}_K + \mathbf{y}_K) & \end{array}$$

  Since H and K are subspaces, $\mathbf{x}_H + \mathbf{y}_H \in H$ and $\mathbf{x}_K + \mathbf{y}_K \in K$. Therefore, $\mathbf{x} + \mathbf{y} \in H + K$.

- Show that, if $\mathbf{x} \in H + K$, then $c\mathbf{x} \in H + K$ for all c.

  SOLUTION: Let $\mathbf{x} \in H + K$. Then

  $$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_K \quad \text{for some } \mathbf{x}_H \in H, \mathbf{x}_K \in K$$

  so that

  $$c\mathbf{x} = c\mathbf{x}_H + c\mathbf{x}_K$$

  Since H and K are subspaces, $c\mathbf{x} \in H$ and $c\mathbf{x}_K \in K$ for all scalars c. Therefore, $c\mathbf{x}$ is in $H + K$ for all c.

When we go to solve Exercise 32, you have to know how to show a vector is in $H \cap K$- Vector $\mathbf{x}$ is in $H \cap K$ only if $\mathbf{x} \in H$ AND $\mathbf{x} \in K$. To start you off, $\vec{0}$ is in $H \cap K$ because $\vec{0} \in H$ and $\vec{0} \in K$ (H, K are subspaces).