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- Question: What is the long term behavior of the orbit?

$$\lim_{t \rightarrow \infty} x_t = \lim_{t \rightarrow \infty} f^{(t)}(x_0)$$

Example 1

Let $x_0 = \frac{1}{2}$, and $f(x) = x^2$. Then:

i	0	1	2	3	...
x_i	1/2	1/4	1/16	1/256	...

This orbit seems to converge to

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Summary of the dynamics using $f(x) = x^2$ and the real line:

If $0 < x_0 < 1$

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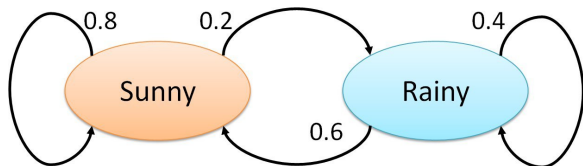
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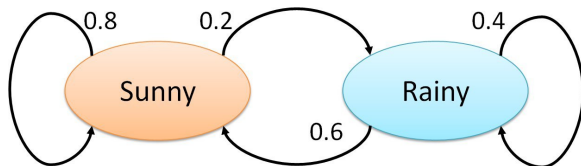
Example 2: Markov Chains

Markov chains can be used to simulate transitions between states using probabilities.



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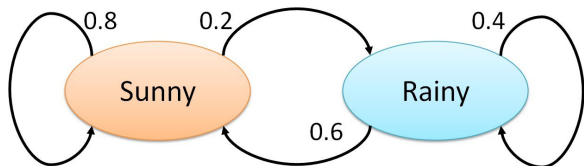
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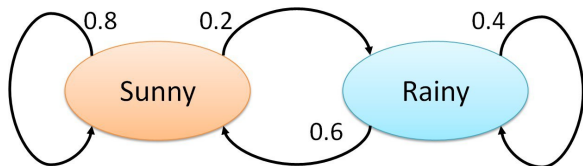


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“If rainy today, the probability of rainy tomorrow is 40%, and the probability of sun is 60%.”

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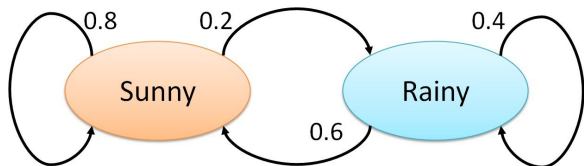
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Using a random choice:

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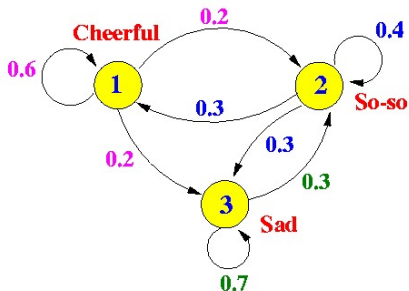
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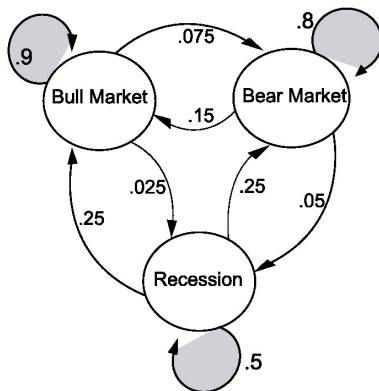
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R S S S S S S S R S R R S S S S S R S S S



You can track your moods...

- C C - - C C C C C - S S - - C C C C C C



You might model the economy...

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people
believe
anything

Markov Chain Poetry

i
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the
dawn
light
before
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something
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Example:

$$M = \begin{array}{c|ccc} & & \text{From} & & \\ & & 1 & 2 & 3 \\ \hline & 1 & 0.7 & 0.3 & 0.1 \\ \text{To:} & 2 & 0.2 & 0.5 & 0.0 \\ & 3 & 0.1 & 0.2 & 0.9 \end{array}$$

The 0.3 entry means there is a probability of 30% of going to state 1 from state 2.

- A stochastic matrix is *regular* if there is a k so that P^k has all non-negative values. (The example is regular, if you check P^2).

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The dynamics: $\mathbf{x}_k = M^k \mathbf{x}_0$.

If 60% of the population is in the city, 40% in the suburbs, we can compute the percentages for next year:

$$\mathbf{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = M\mathbf{x}_0 = \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix} = M^2\mathbf{x}_0 = \begin{bmatrix} 0.565 \\ 0.435 \end{bmatrix}$$

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Definition: x is fixed if $x = f(x)$.

In this context, $\mathbf{x}$ is fixed if

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For example,

$$(M - I)\mathbf{x} = \vec{0} \quad \Rightarrow \quad \begin{bmatrix} 0.95 - 1 & 0.03 \\ 0.05 & 0.97 - 1 \end{bmatrix} \mathbf{x} = \vec{0}$$

$$\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \mathbf{x} = \vec{0}$$

Therefore, if $\mathbf{x} = [c, s]^T$, then:

$$-5c + 3s = 0 \quad \Rightarrow \quad \begin{array}{l} c = 3/5s \\ s = s \end{array}$$

$$= s \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = s_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = s_2 \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

If 3/8 of the population is in the city, 5/8 in the suburbs, then the populations will remain unchanged in time.

HOMEWORK: Exercises 1, 2, 3, 9, 11, 12