STUDY GUIDE: Ch. 1-2

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This review is organized into four main areas: Theory, Analysis, Methods and Models.

We've seen that a differential equation defines a family of functions (an initial value problem defines a specific function). As such, ODEs provide a powerful tool for modeling.

When we solve an ODE, we not only want to get an analytic solution, but we also want to understand graphical analysis (direction fields, phase plots) and we want to be able to analyze the solution- Is the solution unique? What is its behavior over the long term?, etc.

Vocabulary

You should what these terms mean:

differential equation, ordinary differential equation, partial differential equation, order of a differential equation, linear differential equation, equilibrium solution, isocline, direction field, what is a *solution* to a differential equation (and can you verify that a given function solves a DE)

Existence and Uniqueness Theorem

1. Linear: $y' + p(t)y = g(t)$ at (t_0, y_0):

   If p, g are continuous on an interval I that contains t_0, then there exists a unique solution to the initial value problem and that solution is valid for all t in the interval I.

2. General Case: $y' = f(t, y)$, (t_0, y_0):

   (a) If f is continuous on a small rectangle containing (t_0, y_0), then there exists a solution to the initial value problem.

   (b) If $\partial f/\partial y$ is continuous on that small rectangle containing (t_0, y_0), then that solution is unique.

   (c) We can only guarantee that the solution persists on a small interval about (t_0, y_0). To find the full interval, we need to actually solve the initial value problem.

Analysis of Solutions

1. Construct a direction field: Since $y' = f(t, y)$, at each value of (t, y), we can compute the local slope, $y'(t)$. Isoclines can be used to help: An isocline is determined by setting the derivative equal to a constant k and plotting the curve determined by: $k = f(t, y)$.

2. Autonomous DEs: Given that $y' = f(y)$, we can plot (y, y') (this is the phase plot or gives a phase diagram). Be able to translate this graph into the (t, y) plane. For a summary, see the table below:

In Phase Diagram:	In Direction Field:
y intercepts	Equilibrium Solutions
$+$ to $-$ crossing	Stable Equilibrium
$-$ to $+$ crossing	Unstable Equilibrium
$y' > 0$	y increasing
$y' < 0$	y decreasing
y' and df/dy same sign	y is concave up
y' and df/dy mixed	y is concave down

Methods

Our classifications (and solution techniques) were: Linear, Separable, homogeneous (the substitution makes it linear), and Bernoulli (the substitution makes it linear). Autonomous DEs are always separable.

- Linear: $y' + p(t)y = g(t)$. Use the integrating factor: $\mathrm{e}^{\int p(t)\,dt}$
- Homogeneous: $\frac{dy}{dx} = F(y/x)$. Substitute $v = y/x$ (and get the expression for dv/dx as well).
- Bernoulli: $y' + p(t)y = y^n$ Divide by y^n, let $w = y^{1-n}$ and it becomes linear.
- Separation of variables: $y' = f(y)g(t)$ Separate variables: $(1/f(y))\,dy = g(t)\,dt$
- Exact: $M(x,y) + N(x,y)\frac{dy}{dx}$, where $N_x = M_y$.

  Solution: Set $f_x(x,y) = M(x,y)$. Integrate w/r to x. Check that $f_y = N(x,y)$, and add a function of y if necessary.

  The integrating factor for exact equations will not be tested.

Models

1. Construct an autonomous differential equation to model population growth in the standard model and with an environmental carrying capacity. Be able to solve these models analytically (usually requires partial fractions) and graphically.
2. Be able to construct the differential equation corresponding to the tank mixing problem. Be able to solve and analyze the solution (it will be a linear first order equation).
3. Be able to write down Newton's Law of cooling; be able to find the coefficients in the model; solve it analytically, and analyze the behavior of the solutions.
4. Know the model for an object in free fall: $mv' = mg - kv$. What assumptions did we make to write this? For these problems, I will give you the constant(s) for g.

Construct A Differential Equation

It is useful to construct your own differential equations so that you get a better feel for how they come about. This can be done through the modeling process, or directly. Be sure to try out the sample problems.