#1 $y'' + 2y' + 2y = \delta(t-\pi)$, $y(0) = 1$, $y'(0) = 0$.

$$s^2Y - (sy(0)) - y'(0) + 2(sY - y(0)) + 2Y = e^{-\pi s}$$

$$(s^2+2s+2)\,Y = e^{-\pi s} + 2 + s$$

So $Y = \dfrac{e^{-\pi s}}{s^2+2s+2} + \dfrac{s+2}{s^2+2s+2}$.

To invert, consider:

$$\frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\overset{\mathcal{L}^{-1}}{=} e^{-t}\cos(t) + e^{-t}\sin(t)$$

For the other one, $e^{-\pi s}H(s)$, we know $h(t) = e^{-t}\sin t$,

so the inverse of this part is $u_\pi(t)h(t-\pi)$.

Overall,

$$y(t) = e^{-t}\cos(t) + e^{-t}\sin(t) + u_\pi(t)\,e^{-(t-\pi)}\sin(t-\pi)$$

Plot in Maple.

#2 $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$ $y(0)=0$, $y'(0)=0$

$$(s^2+4)Y = e^{-\pi s} - e^{-2\pi s} \Rightarrow Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} = (e^{-\pi s} - e^{-2\pi s})\,H(s)$$

So $H(s) = \frac{1}{2}\frac{2}{s^2+4}$, $h(t) = \frac{1}{2}\sin(2t)$ and the soln is:

$$y(t) = \tfrac{1}{2}u_\pi(t)\sin(2(t-\pi)) - \tfrac{1}{2}u_{2\pi}(t)\sin(2(t-2\pi))$$

Analyze this in parts: $\sin(2t)$ is periodic w/ period π,

so $\sin(2(t-\pi)) = \sin(2t) = \sin(2(t-2\pi))$.

Therefore, breaking up the solution in parts:

$$y(t) = \begin{cases} 0 & t < \pi \\ \frac{1}{2}\sin(2t) & \pi \le t < 2\pi \\ 0 & t \geq 2\pi. \end{cases}$$

#3. $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$, $y(0)=0$, $y'(0)=\frac{1}{2}$

$$s^2Y - \frac{1}{2} + 3sY + 2Y = e^{-5s} + \frac{e^{-10s}}{s}$$

$$(s^2+3s+2)Y = \frac{e^{-10s}}{s} + e^{-5s} + \frac{1}{2}$$

$$Y = \frac{e^{-10s}}{s(s^2+3s+2)} + \frac{e^{-5s}}{s^2+3s+2} + \frac{1}{2}\frac{1}{s^2+3s+2}$$

We note that:

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

and

$$\frac{1}{s^2+3s+2} = -\frac{1}{s+2} + \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} -e^{-2t} + e^{-t}$$

Therefore,

$$y(t) = +\frac{1}{2}(-e^{-2t} + e^{-t}) + u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right) + u_{10}(t)\left(\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right)$$

(Plot in Maple)

#5 $\quad y'' + 2y' + 3y = \sin t + \delta(t - 3\pi), \quad y(0) = 0, \quad y'(0) = 0$

$$(s^2 + 2s + 3)Y = \frac{1}{s^2+1} + e^{-3\pi s} \Rightarrow Y = \frac{1}{(s^2+1)(s^2+2s+3)} + \frac{e^{-3\pi s}}{(s^2+2s+3)}$$

Setting up partial fractions:

$$\frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 3} = \frac{1}{(s^2+1)(s^2+2s+3)}$$

We found that:

$$\frac{1}{(s^2+1)(s^2+2s+3)} = \frac{1}{4}\frac{s+1}{s^2+2s+3} - \frac{1}{4}\frac{s-1}{s^2+1}$$

$$= \frac{1}{4}\frac{s+1}{s^2+2s+1+2} - \frac{1}{4}\left(\frac{s}{s^2+1} - \frac{1}{s^2+1}\right)$$

$$= \frac{1}{4}\frac{s+1}{(s+1)^2+2} - \frac{1}{4}\left(\frac{s}{s^2+1} - \frac{1}{s^2+1}\right)$$

$$\mathcal{L}^{-1} \Rightarrow \frac{1}{4}e^{-t}\cos(\sqrt{2}t) - \frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)$$

And

$$H(s) = \frac{1}{s^2+2s+3} = \frac{1}{(s+1)^2+2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} \xrightarrow{\mathcal{L}^{-1}} h(t) = \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$$

Put them together:

$$y(t) = \frac{1}{4}e^{-t}\cos(\sqrt{2}t) - \frac{1}{4}\cos t + \frac{1}{4}\sin t + u_{3\pi}(t)\cdot\frac{1}{\sqrt{2}}e^{-(t-3\pi)}\cdot\sin(\sqrt{2}(t-3\pi))$$

(plot in Maple)

#7 $\quad y'' + y = \delta(t-2\pi)\cos(t) \quad y(0)=0, \quad y'(0)=1$

$$s^2 Y - 1 + Y = \mathcal{L}\left(\delta(t-2\pi)\cos(t)\right).$$

Question: What is $\mathcal{L}(\delta(t-c)f(t))$?

By def, $\mathcal{L}(\delta(t-c)f(t)) = \int_0^\infty e^{-st}\delta(t-c)f(t)\,dt = e^{-sc}\underbrace{f(c)}_{\text{constant.}}$

In our case,

$$\mathcal{L}\left(\delta(t-2\pi)\cos(t)\right) = e^{-2\pi s}\cos(2\pi) = e^{-2\pi s}$$

Therefore, the DE becomes:

$$(s^2+1)Y = e^{-2\pi s} + 1$$

$$Y = \frac{e^{-2\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

$$y(t) = \sin(t) + u_{2\pi}(t)\sin(t-2\pi) = \sin(t) + u_{2\pi}(t)\sin(t)$$

$$y(t) = \begin{cases} \sin(t), & t < 2\pi \\ 2\sin(t) & t > 2\pi. \end{cases}$$

(Plot in Maple)

#14 We might go ahead and do this generally.

$$y'' + \gamma y + y = \delta(t-1) \qquad \text{zero ICs}$$

$$(s^2 + \gamma s + 1)\, Y = e^{-s}$$

$$Y = \frac{e^{-s}}{s^2+\gamma s+1} = e^{-s} H(s), \text{ with } H(s) = \frac{1}{s^2+\gamma s+1}.$$

In determining if this is factorable, find the roots of the denominator: $s^2+\gamma s+1=0 \Rightarrow s = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2-4}}{2}$

In looking at the question, $\gamma = 1/2$ and $\downarrow 0$, so in that case, we have complex roots. Therefore we could complete the square:

$$\frac{1}{s^2+\gamma s+1} = \frac{1}{s^2+\gamma s+\frac{\gamma^2}{4}+1-\frac{\gamma^2}{4}} = \frac{1}{(s+\frac{\gamma}{2})^2+\left(\frac{4-\gamma^2}{4}\right)}$$

$$\frac{1}{\sqrt{\left(\frac{4-\gamma^2}{4}\right)}} \frac{\sqrt{\left(\frac{4-\gamma^2}{4}\right)}}{\left[(s+\frac{\gamma}{2})^2+\left(\frac{4-\gamma^2}{4}\right)\right]}$$

$$h(t) = \frac{2}{\sqrt{4-\gamma^2}} e^{-\frac{\gamma}{2}t} \sin\left(\left(\frac{\sqrt{4-\gamma^2}}{2} t\right)\right)$$

and the solution will be $u_1(t)\, h(t-1)$.

t_1 will be when $\sqrt{\frac{4-\gamma^2}{4}}\, t_1 = \frac{\pi}{2} \Rightarrow t_1 = \frac{\pi}{\sqrt{4-\gamma^2}}$ (then add 1)

(see Maple for plot)

#15 $y'' + \gamma y' + y = k\,\delta(t-1)$, zero ICs

We've solved this IVP in 14:

$$y(t) = k\,u_1(t)\,h(t-1)$$

The peak value occurred @ $t_1 = 1 + \frac{\pi}{\sqrt{4-\gamma^2}}$. At

that value, $h(t-1) = \frac{2}{\sqrt{4-\gamma^2}}\, e^{-\frac{\gamma}{2}\left(\frac{\pi}{\sqrt{4-\gamma^2}}\right)}$

so $k = e^{\frac{\gamma}{2}\left(\frac{\pi}{\sqrt{4-\gamma^2}}\right)} \sqrt{4-\gamma^2}$.

As $\gamma \to 0$, these expressions are all continuous.

In particular, $k \to 2$.

#17 $y'' + y = \sum_{k=1}^{20} \delta(t - k\pi)$, zero ICs

$$Y = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2+1} \Rightarrow y(t) = \sum_{k=1}^{20} u_{k\pi}(t) \sin(t - k\pi).$$

Graphically, at time π, the sine curve begins. At half the period, the next function comes in. In that case, we take $\sin(t-\pi) + \sin(t-2\pi) = -\sin(t) + \sin(t) = 0$, and at the next multiple of π, the pattern begins again. (see Maple Plot)

#18 $f(t) = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi)$

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} u_{k\pi}(t) \sin(t - k\pi)$$

We begin the same way, but now the "taps" go with the system: Initially, we have zero, then at π, a sine wave starts. At $t = 2\pi$, the next piece comes in:

$$\sin(t-\pi) - \sin(t-2\pi) = -3\sin(t)$$

The amplitude continues to grow until it reaches 20 at 20π.

#19 Harder to determine - See the Maple plot.