Review Questions: Exam 3

1. Define the new functions of the section: $u_c(t)$ and $\delta(t-c)$ (Note: the Dirac function should be defined as a certain limit). Define the new operation, $(f*g)(t)$

2. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)
   $$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 6-t, & t \ge 2 \end{cases}$$

   (b)
   $$f(t) = \begin{cases} \mathrm{e}^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

3. Check your answers to Problem 2 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

4. Determine the Laplace transform:

   (a) $t^2\mathrm{e}^{-9t}$
   (b) $\mathrm{e}^{2t} - t^3 - \sin(5t)$
   (c) $t^2y'(t)$ (in terms of $Y(s)$)
   (d) $\mathrm{e}^{3t}\sin(4t)$
   (e) $\mathrm{e}^t\delta(t-3)$
   (f) $t^2u_4(t)$

5. Find the inverse Laplace transform:

   (a) $\dfrac{2s-1}{s^2-4s+6}$
   (b) $\dfrac{7}{(s+3)^3}$
   (c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)}$
   (d) $\dfrac{3s-1}{2s^2-8s+14}$
   (e) $\left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)\dfrac{1}{s^2+s-6}$

6. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$
   (b) $y'' + 9y = 10\mathrm{e}^{2t}$, $y(0) = -1$, $y'(0) = 5$
   (c) $y'' - 4y' + 4y = t^2\mathrm{e}^t$, $y(0) = 0$, $y'(0) = 0$

7. Solve the given initial value problems using Laplace transforms:

   (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.
   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$
   (c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
   (d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$
   (e) $y'' + y = \displaystyle\sum_{k=1}^{\infty}\delta(t-2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

8. Short Answer:

   (a) $\int_0^\infty \sin(3t)\delta(t-\frac{\pi}{2})\,dt =$_____

   (b) If $y''+2y'+3y=0$ and $y(0)=1$, $y'(0)=-1$, compute $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$.

   (c) Using your previous result, give the Taylor expansion of the solution to the differential equation using at least 5 terms.

   (d) If $y'(t)=\delta(t-c)$, what is $y(t)$?

   (e) What is the expected radius of convergence for the series expansion of $f(x)=1/(x^2+2x+5)$ if the series is based at $x_0=1$?

   (f) Use Laplace transforms to solve for $F(s)$, if

$$f(t)+2\int_0^t \cos(t-x)f(x)\,dx = \mathrm{e}^{-t}$$

   (So only solve for the transform of $f(t)$, don't invert it back).

   (g) In order for the Laplace transform of f to exist, f must be _____

   (h) Can we assume that the solution to: $y''+p(x)y'+q(x)y=u_3(x)$ is a power series?

   (i) Is $x=0$ an ordinary point for the differential equation: $y''+\sqrt{x}\,y'+xy=x^2$?

9. More on Laplace Transforms:

   (a) Your friend tells you that the solutions to the IVPs:

$$y''+2y'+y=0,\ \ y(0)=0,\ y'(0)=1 \qquad \text{and} \qquad y''+2y'+y=\delta(t)\ \ y(0)=0,\ y'(0)=0$$

   are exactly the same. Are they really? Explain.

   (b) Let $f(t)=t$ and $g(t)=u_2(t)$.

      i. Use the Laplace transform to compute $f*g$.

      ii. Verify your answer by computing $f*g$ using the definition.

10. Find the recurrence relation between the coefficients for the power series solutions to the following:

    (a) $2y''+xy'+3y=0$, $x_0=0$.

    (b) $(1-x)y''+xy'-y=0$, $x_0=0$

    (c) $y''-xy'-y=0$, $x_0=1$

11. Find the first 5 terms of the power series solution to $\mathrm{e}^x y''+xy=0$ if $y(0)=1$ and $y'(0)=-1$.

12. Find the radius of convergence for the following series:

    (a) $\sum_{n=1}^{\infty}\sqrt{n}x^n$

    (b) $\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$

    (c) $\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n}$