

## Examples for 5.2 and 5.3

### Exercise 3, Section 5.2

Find the recurrence relation and the first four terms of two linearly independent solutions to:

$$y'' - xy' - y = 0 \quad x_0 = 1$$

Substitute the sum:  $y = \sum a_n(x-1)^n$  in for  $y, y'$  and  $y''$ :

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

To simplify the second term, we need to be able to bring in an  $(x-1)$ , not just  $x$ . Therefore, the middle sum becomes two sums:

$$\begin{aligned} x \sum_{n=1}^{\infty} na_n(x-1)^{n-1} &= (x-1) \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \\ &= \sum_{n=1}^{\infty} na_n(x-1)^n + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \end{aligned}$$

And now there are 4 sums, and they simplify to:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Our goal now is to collect all the sums together as a single series of the form:

$$\sum_k \left( \quad \right) (x-1)^k = 0$$

To start this process, we want each sum to begin with the same power. The first, third and fourth sums begin with  $(x-1)^0$ , but the second sum does not. In this case, we just need to make the index start at zero instead of 1 (this does not change the sum, since the first term would be 0):

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Now we put everything into the same sum. We make the following substitutions (in order):

$$\begin{array}{cccc} k = n - 2 & k = n & k = n - 1 & k = n \\ n = k + 2 & n = k & n = k + 1 & n = k \end{array}$$

To get:

$$\sum_{k=0}^{\infty} ((k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k) (x-1)^k = 0$$

The recurrence relation is found by setting the coefficients to zero (recall that if a polynomial is zero for all values of  $x$ , then every coefficient must be zero):

$$((k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k = 0 \quad \text{for } k = 0, 1, 2, 3, \dots$$

Solve for  $a_{k+2}$  to get the recurrence relation:

$$a_{k+2} = \frac{1}{k+2}(a_k + a_{k+1}) \quad \text{for } k = 0, 1, 2, 3, \dots$$

Now we want to find the first four terms of two linearly independent solutions. Recall from Chapter 3 that we can do this by taking  $y_1$  and  $y_2$  as solutions to the ODE, with initial conditions (respectively). That is,  $y_1$  solves the DE with initial conditions:

$$y(1) = a_0 = 1 \quad y'(1) = a_1 = 0$$

And  $y_2$  solves the DE with initial conditions:

$$y(1) = a_0 = 0 \quad y'(1) = a_1 = 1$$

We do this for  $y_1$  in the first column,  $y_2$  in the second:

$$\begin{array}{ll} k=0 & a_2 = \frac{1}{2}(a_0 + a_1) = \frac{1}{2} \\ k=1 & a_3 = \frac{1}{3}(a_1 + a_2) = \frac{1}{6} \\ k=2 & a_4 = \frac{1}{4}(a_2 + a_3) = \frac{1}{6} \\ k=3 & a_5 = \frac{1}{5}(a_3 + a_4) = \frac{1}{15} \end{array} \quad \text{and} \quad \begin{array}{ll} k=0 & a_2 = \frac{1}{2}(a_0 + a_1) = \frac{1}{2} \\ k=1 & a_3 = \frac{1}{3}(a_1 + a_2) = \frac{1}{2} \\ k=2 & a_4 = \frac{1}{4}(a_2 + a_3) = \frac{1}{4} \\ k=3 & a_5 = \frac{1}{5}(a_3 + a_4) = \frac{3}{20} \end{array}$$

Therefore,

$$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \dots$$

and

$$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots$$

**Exercise 2, Section 5.3**

Determine the values of the first, second, third and fourth derivatives of the solution at the given value of  $x_0$ . (Added: Write the corresponding partial Taylor series):

$$y'' + \sin(x)y' + \cos(x)y = 0 \quad y(0) = 0 \quad y'(0) = 1$$

In this case, it is easy to solve for  $y''$  and differentiate:

$$y'' = -\sin(x)y' - \cos(x)y$$

Evaluate at  $x = 0$ :  $y''(0) = -0 \cdot 1 - 1 \cdot 0 = 0$ . Continue differentiating:

$$y''' = -\cos(x)y' - \sin(x)y'' + \sin(x)y - \cos(x)y' = -2\cos(x)y' - \sin(x)y'' + \sin(x)y$$

Evaluate at  $x = 0$ :  $y'''(0) = -2 \cdot 1 \cdot 1 - 0 \cdot 0 + 0 \cdot 0 = -2$  Continuing:

$$y^{iv} = 2\sin(x)y' - 2\cos(x)y'' - \cos(x)y'' - \sin(x)y''' + \cos(x)y + \sin(x)y'$$

Simplify:

$$= 3\sin(x)y' - 3\cos(x)y'' - \sin(x)y''' + \cos(x)y$$

Evaluate at  $x = 0$ ;

$$y^{iv}(0) = 0 \cdot 0 - 3 \cdot 0 - 0 \cdot -2 + 1 \cdot 0 = 0$$

Writing the power series solution using the Taylor formula, so far we have:

$$y(x) = 0 + 1 \cdot x + 0 \cdot x^2 - \frac{2}{3!}x^3 + 0 \cdot x^4 + \dots$$

If we want more constants, we could compute them. For fun, here is the expansion up to the 8th power:

$$y(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{59}{2520}x^7 + \dots$$

### Extra Example, Section 5.3

Determine a lower bound for the radius of convergence of the series solutions about each given point  $x_0$  for the given differential equation:

$$(x^2 - x - 2)y'' + x^2y' - 4y = 0 \quad x_0 = 3 \quad x_0 = 6$$

SOLUTION: Dividing by the polynomial in front of  $y''$ , we have two functions that should be analytic about  $x_0$ . Notice that the polynomial factors:

$$\frac{x^2}{(x-2)(x+1)} \quad \frac{4}{(x-2)(x+1)}$$

We notice that both functions have singularities at  $x = 2$  and  $x = -1$  (these are points at which the denominator is zero).

If the power series is based at  $x_0 = 3$ , there is a distance of  $|3 - 2| = 1$  to the point  $x = 2$  and a distance of  $|3 - (-1)| = 4$  to the point  $x = -1$ . We take the smaller of these as the radius.

Conclusion: The lower bound for the radius of convergence is 2.

If the power series is based at  $x_0 = 6$ , the distance between the two “bad points” is 4 and 7, respectively. Take the lower bound for the radius to be 4.

### Modify the example slightly:

If we change the polynomial in front to:

$$(x^2 + 2x + 3)y'' + x^2y' - 4y = 0 \quad x_0 = 3$$

Using the quadratic formula, we see that  $x^2 + 2x + 3 = 0$  if  $x = -1 \pm \sqrt{2}i$ . In the complex plane, we would plot these as the points  $(-1, \sqrt{2})$  and  $(-1, -\sqrt{2})$ .

To find the radius of convergence, we need to find the smaller distance of these points to our point,  $x_0 = 3 + 0i$  or  $(3, 0)$ .

The distance formula for  $(-1, \sqrt{2})$  gives:

$$\sqrt{(3 - (-1))^2 + (\sqrt{2} - 0)^2} = \sqrt{18} = 3\sqrt{2}$$

If you compute the distance to the other point, you'll get the same thing.

Conclusion: The lower bound for the radius of convergence is  $3\sqrt{2}$