

# Solutions to the Review Questions

## Short Answer/True or False

1. True or False, and explain:

- (a) If  $y' = y + 2t$ , then  $0 = y + 2t$  is an equilibrium solution.

False: This is an isocline associated with a slope of zero, and furthermore,  $y = -2t$  is not a solution.

- (b) Let  $\frac{dy}{dt} = 1 + y^2$ . The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all  $t$ . (If true, say why. If False, solve the DE).

False. The E & U Theorem tells that a unique solution will exist for any initial condition (since  $1 + y^2$  and  $2y$  are continuous everywhere), but it does not say on what interval the solution will exist. For example, if we take  $y(0) = y_0$  and solve, we get:

$$\int \frac{dy}{1+y^2} = \int dt \Rightarrow \tan^{-1}(y) = t + C \Rightarrow C = \tan^{-1}(y_0)$$

Therefore,

$$y = \tan(t + \tan^{-1}(y_0))$$

where

$$-\frac{\pi}{2} < t + \tan^{-1}(y_0) < \frac{\pi}{2}$$

(so that the tangent function is invertible).

- (c) Let  $y' = f(y)$ . It is possible to have two stable equilibrium with no other equilibrium between them (you may assume  $df/dy$  is continuous).

False. Suppose there is a stable equilibrium at  $y = a$  and at  $y = b$ . At  $y = a$ ,  $f(y)$  must cross the  $y$ -axis from a positive value to a negative (draw a sketch), and similarly, at  $y = b$ ,  $f$  must cross from positive to negative. Therefore,  $f$  is going from a slightly negative number to the right of  $y = a$  to a slightly positive number to the left of  $y = b$ . Since we're assuming  $f$  is continuous, the Intermediate Value Theorem states that  $f$  must cross the  $y$ -axis between  $a$  and  $b$ , and in fact, this crossing must be from a negative to a positive number (which is an unstable equilibrium).

- (d) All autonomous equations are separable.

True. Any autonomous equation can be written as  $y' = f(y) \cdot 1$ , which is separable and  $\int \frac{dy}{f(y)} = \int dt$ .

- (e) All linear (first order) equations are Bernoulli.

True- It is possible to categorize them this way:

$$y' + p(t)y = g(t)y^0$$

but we typically will not write them this way (so you could argue "false" as well).

- (f) All separable equations are exact.

True. If the equation is separable, then  $y' = f(y)g(x)$ , which can be written:

$$\frac{1}{f(y)} \frac{dy}{dx} = g(x) \Rightarrow -g(x) + \frac{1}{f(y)} \frac{dy}{dx} = 0$$

Now, if  $M(x, y) = -g(x)$ , then  $M_y = 0$ , and  $N(x, y) = 1/f(y)$  means  $N_x = 0$ .

2. The Existence and Uniqueness Theorems:

- Linear:  $y' + p(t)y = g(t)$  at  $(t_0, y_0)$ :

If  $p, g$  are continuous on an interval  $I$  that contains  $t_0$ , then there exists a unique solution to the initial value problem and that solution is valid for all  $t$  in the interval  $I$ .

- General Case:  $y' = f(t, y)$ ,  $(t_0, y_0)$ :

Let the functions  $f$  and  $f_y$  be continuous in some open rectangle  $R$  containing the point  $(t_0, y_0)$ . Then there exists an interval about  $t_0$ ,  $(t_0 - h, t_0 + h)$  contained in  $R$  for which a unique solution to the IVP exists.

*Side Remark 1:* To determine such a time interval, we must solve the DE.

*Side Remark 2:* We broke out the theorem in class into two components (existence and uniqueness). You can use either the theorem there or as it stated above.

3. To solve  $y' = y^{1/3}$ , we separate variables:

$$y^{-1/3} dy = dt$$

Before going further, it is good practice to note that the previous step is valid, *as long as*  $y \neq 0$ . The case that  $y = 0$  can be taken separately- In fact, we see that  $y(t) = 0$  is an equilibrium solution that satisfies the initial condition.

Going on, we integrate:

$$\frac{3}{2}y^{2/3} = t + C_1 \quad \Rightarrow \quad y^{2/3} = \frac{2}{3}t + C_2 \quad \Rightarrow \quad y = \left(\frac{2}{3}t + C_2\right)^{3/2}$$

We can solve for  $C_2$  using the initial condition:  $0 = C_2$ , so that

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

We can verify that this is indeed a solution by substituting it back into the DE (not necessary; just a way of double-checking yourself):

$$y' = \frac{3}{2} \left(\frac{2t}{3}\right)^{1/2} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{1/2}$$

And on the other hand,

$$y^{1/3} = \left[\left(\frac{2t}{3}\right)^{3/2}\right]^{1/3} = \left(\frac{2t}{3}\right)^{1/2}$$

Therefore, this is indeed a second solution to the IVP.

Of course, the Existence and Uniqueness Theorem cannot be “violated” since it is a theorem, but in this case, the *hypotheses* are not satisfied:

$$y' = f(t, y) \quad \Rightarrow \quad f(t, y) = y^{1/3}$$

In this case,  $f$  is continuous at  $(0, 0)$  but  $\partial f / \partial y$  is not.

### Solve:

1.  $\frac{dy}{dx} = \frac{x^2 - 2y}{x}$

Linear:  $y' + \frac{2}{x}y = x$  Solve with an integrating factor of  $x^2$  to get:

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2.  $(x + y) dx - (x - y) dy = 0$ . Hint: Let  $v = y/x$ .

Given the hint, rewrite the DE:

$$\frac{dy}{dx} = \frac{x + y}{x - y} = \frac{1 + (y/x)}{1 - (y/x)} = \frac{1 + v}{1 - v}$$

With the substitution  $xv = y$ , we get the substitution for  $dy/dx$ :

$$v + xv' = y'$$

So that the DE becomes:

$$v + xv' = \frac{1+v}{1-v} \Rightarrow xv' = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

The equation is now separable:

$$\frac{1-v}{1+v^2} dv = \frac{1}{x} dx \Rightarrow \int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \ln|x| + C$$

Therefore,

$$\tan^{-1}(v) - \frac{1}{2} \ln(1+v^2) = \ln|x| + C$$

Lastly, back-substitute  $v = y/x$ .

3.  $\frac{dy}{dx} = \frac{2x+y}{3+3y^2-x} \quad y(0) = 0.$

This is exact. The solution is, with  $y(0) = 0$ ,

$$-x^2 - xy + 3y + y^3 = 0$$

4.  $\frac{dy}{dx} = -\frac{2xy+y^2+1}{x^2+2xy}$

This is exact. The solution is:  $x^2y + xy^2 + x = c$

5.  $\frac{dy}{dt} = 2 \cos(3t) \quad y(0) = 2$

This is linear and separable.  $y(t) = \frac{2}{3} \sin(3t) + 2$ , and the solution is valid for all time.

6.  $y' - \frac{1}{2}y = 0 \quad y(0) = 200.$  State the interval on which the solution is valid.

This is linear and separable. As a linear equation, the solution will be valid on all  $t$  (since  $p(t) = -\frac{1}{2}$ ).

The solution is  $y(t) = 200e^{(1/2)t}$

7. This is separable (or Bernoulli):

$$\int y^{-2} dy = \int (1-2x) dx \Rightarrow -\frac{1}{y} = x - x^2 + C$$

Put in the initial condition (IC):  $6 = 0 + C$ . Now finish solving explicitly:

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

The solution is valid on the interval  $(-2, 3)$ .

8.  $y' - \frac{1}{2}y = e^{2t} \quad y(0) = 1$

This is linear (but not separable).  $y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{(1/2)t}$

9.  $y' = \frac{1}{2}y(3-y)$

Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)} dy = \frac{1}{2} \int dt$$

Simplify your answer for  $y$  by dividing numerator and denominator appropriately to get:

$$y(t) = \frac{3}{(1/A)e^{-(3/2)t} + 1}$$

10.  $\sin(2t) dt + \cos(3y) dy = 0$

Separable (and/or exact):  $\frac{-1}{2} \cos(2t) + \frac{1}{3} \sin(3y) = C$

11.  $y' = xy^2$

Separable:  $y = \frac{1}{-(1/2)x^2 - C}$

12.  $\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}$  (Hint: Think Bernoulli)

SOLUTION: Multiply by  $y$  to get

$$yy' - \frac{3}{2x}y^2 = 2x$$

So (following what we did for the general Bernoulli eqn), let  $v = y^2$ , and therefore  $v' = 2yy'$ . Multiply by 2 to get the right form, then substitute

$$2yy' - \frac{3}{x}y^2 = 4x \Rightarrow v' - \frac{3}{x}v = 4x$$

Now it is a standard linear DE. Solving, we get  $v = -4x^3 + Cx^3$ , and

$$y^2 = -4x^3 + Cx^3$$

(We'll leave in implicit form).

13.  $yy'' = (y')^2$  With the hint, we need to find a substitution for  $y''$  (and then leave the equation as a DE with unknown function  $p$  with variable  $y$ ):

$$\frac{dy}{dt} = p(y) \Rightarrow \frac{d^2y}{dt^2} = \frac{dp}{dy} \cdot \frac{dy}{dt} = \frac{dp}{dy}p(y)$$

Now, substitute in the expressions and divide by  $yp(y)$ :

$$y \frac{dp}{dy} p(y) = (p(y))^2 \Rightarrow \frac{dp}{dy} = \frac{1}{y}p(y) \Rightarrow \int \frac{1}{p} dp = \int \frac{1}{y} dy \Rightarrow \ln |p| = \ln |y| + C \Rightarrow p = Ay$$

where  $A$  is a constant of integration. Now, substitute again with  $p = y'$ , and we get:

$$y' = Ay \Rightarrow y = Pe^{At}$$

where  $P$  is another constant of integration.

We can verify our answer as well:  $y' = APe^{At}$ , and  $y'' = A^2Pe^{At}$  so that

$$yy'' = Pe^{At} \cdot A^2Pe^{At} = A^2P^2e^{2At}$$

and this expression is also  $(y')^2$ .

14.  $y' + 2y = g(t)$  with  $y(0) = 0$  and  $g(t) = 1$  on  $0 \leq t \leq 1$  and zero elsewhere.

SOLUTION: This is similar to Exercise 33, Section 2.4. In this case, we go ahead and solve starting at time 0:

$$y' + 2y = 1 \Rightarrow y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

and this is valid for  $0 \leq t \leq 1$ . When we hit  $t = 1$ , the dynamics change to:

$$y' + 2y = 0 \Rightarrow y(t) = Pe^{-2t}$$

Now we will typically choose the constants so that  $y$  is continuous. Therefore, using our previous function,  $y(1) = (1 - e^{-2})/2$ , and our current function:  $y(1) = Pe^{-2}$ , or

$$P = \frac{e^2 - 1}{2}$$

Therefore, the overall solution to the DE would be:

$$y(t) = \begin{cases} (1 - e^{-2t})/2 & \text{if } 0 \leq t \leq 1 \\ ((e^2 - 1)/2)e^{-2t} & \text{if } 1 > t \end{cases}$$

Just for fun, the direction field and solution curve are plotted in Figure 1.

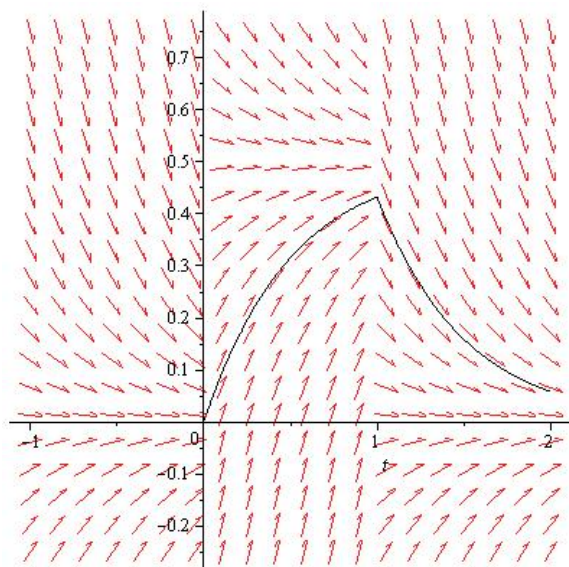


Figure 1: Direction field and solution curve for Exercise 14. Note how the solution approaches one equilibrium until  $g(t)$  changes, then it goes to the new equilibrium.

### Misc.

1. Construct a linear first order differential equation whose general solution is given by:

(a)  $y(t) = t - 3 + \frac{C}{t^2}$

SOLUTION: Construct  $y'$ . The idea will be to produce a linear DE. Therefore, we need to construct  $y'$  and compare it to  $y$ :

$$y' = 1 - 2Ct^{-3}$$

Add this to some multiple ( $t$ 's are allowed) of  $y$  to get rid of the arbitrary constant. In this case,

$$y' + \frac{2}{t}y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

or,

$$ty' + 2y = 3t - 6$$

(b)  $y(t) = 2 \sin(3t) + Ce^{-2t}$

SOLUTION: Same idea as before. Try to get  $y$  and  $y'$  together in such a way as to cancel out the arbitrary  $C$ :

$$y' = 6 \cos(3t) - 2Ce^{-2t}$$

so that:  $y' + 2y = 4 \sin(3t) + 6 \cos(3t)$ .

2. Construct a linear first order DE so that all solutions tend to  $y = 3$  as  $t \rightarrow \infty$ .

SOLUTION: One way to think about it is to construct  $y' = ay + b$ , which is also autonomous- In fact, we are saying that we need a line in the plane, through  $y = 3$  so that the equilibrium is stable- That is any line through  $(0, 3)$  with a negative slope. For example,  $y' = -y + 3$

3. Suppose we want to construct a population model so that there is a logistic population growth-

- (a) That is, there is an environmental carrying capacity of 100. Construct an appropriate (autonomous) model.

SOLUTION: In the  $(y, y')$  plane, we are looking at an upside down parabola that goes through  $y = 0$  and  $y = 100$ . One model is therefore

$$y' = y(100 - y)$$

- (b) Using your model, now assume there is a continuous “harvest” (of  $k$  units per time period). How does that effect the model- In particular, is there a critical value of  $k$  over which the population will be extinct? If so, find it.

SOLUTION: This subtracts  $k$  from our population:

$$y' = y(100 - y) - k$$

We see that for some value of  $k$ , the vertex of the upside parabola is exactly on the  $y$ -axis. We can solve this by completing the square- That is, we should be able to write  $y'$  as a perfect square:

$$-y^2 + 100y + k \rightarrow -(y^2 - 100y + 50^2) = -(y - 50)^2$$

Therefore,  $k = 50^2 = 2500$ . This is the maximum allowable harvest before the ecosystem collapses.

4. Suppose we have a tank that contains  $M$  gallons of water, in which there is  $Q_0$  pounds of salt. Liquid is pouring into the tank at a concentration of  $r$  pounds per gallon, and at a rate of  $\gamma$  gallons per minute. The well mixed solution leaves the tank at a rate of  $\gamma$  gallons per minute.

Write the initial value problem that describes the amount of salt in the tank at time  $t$ , and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \quad Q(0) = Q_0$$

The solution is:

$$Q = rM + (Q_0 - rM)e^{-(\gamma/M)t}$$

5. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on  $Q_0$ ? Does this make sense?

Note that the differential equation for  $Q$  is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as  $t \rightarrow \infty$  and see that  $Q \rightarrow rM$ . This does not necessarily depend on  $Q_0$ ; if  $Q_0$  starts at equilibrium,  $rM$ , then  $Q$  is constant.

It does make sense. The incoming concentration of salt is  $r$  pounds per gallon, so we would expect the long term concentration to be the same,  $rM/M = r$ .

6. Modify problem 5 if:  $M = 100$  gallons,  $r = 2$  and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if  $Q_0 = 50$ .

$$\frac{dQ}{dt} = 4 - \frac{3}{100-t}Q \quad Q(0) = 50$$

This goes from being autonomous to linear. In this case, use an integrating factor,

$$e^{\int p(t) dt} = e^3 \int \frac{1}{100-t} dt = e^{-3 \ln |100-t|} = (100+t)^{-3} \quad t > 100$$

Going back to the DE

$$\left( \frac{Q}{(100-t)^3} \right) = 4(100-t)^{-3} \Rightarrow Q = 2(100-t) + C(100-t)^3$$

Continuing, we get:

$$Q(t) = 2(100-t) - \frac{150}{100^3}(100-t)^3$$

7. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of  $\frac{1}{2}v$ , find the initial value problem (and solve it) for the velocity at time  $t$ .

The general model is:  $mv' = mg - kv$ . In this case,  $m = 1$ ,  $g = 9.8$  and  $k = 1/2$ . Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

Solve it:

$$v(t) = 19.6 \left( 1 - e^{-(1/2)t} \right)$$

8. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of  $k$  dollars per month, set up a model for how much you owe at time  $t$  in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

SOLUTION: First, notice that the units of time are mixed- The interest rate is an annual rate, but  $k$  is in dollars per month. We should first decide on what the units of time should be. The answer below will assume that we are working with time in *years*, so that the annual payments are  $12k$  dollars per year (in a continuous fashion).

Therefore, the model is  $S' = rS - 12k$ , where  $S$  will be the amount owing,  $r$  is the annual interest rate and  $k$  is the rate for the continuous payment (per month). Then using  $S(0) = S_0$ , we can write the solution as

$$S(t) = \frac{12k}{r} + \left( S_0 - \frac{12k}{r} \right) e^{rt}$$

Substituting in the values for  $r$  and  $S_0$ , and  $t = 10$ , we can solve the equation for  $k$ :

$$0 = 240k + (10000 - 240k)e^{\frac{1}{2}}$$

*Extra:* If we go ahead and solve, we get a monthly payment rate of  $k \approx \$105.90$ .

9. Consider the sketch below of  $F(y)$ , and the differential equation  $y' = F(y)$ .

- (a) Find and classify the equilibrium.

SOLUTION: From the sketch given,  $y = 0$  is asymptotically stable and  $y = 1$  is semistable.

- (b) Find intervals (in  $y$ ) on which  $y(t)$  is concave up.

SOLUTION: Examine the intervals  $y < 0$ ,  $0 < y < 1/3$ ,  $1/3 < y < 1$  and  $y > 1$  separately. The function  $y$  will be concave up when  $dF/dy$  and  $F$  both have the same sign- This happens when  $F$  is either increasing and positive (which happens nowhere) or decreasing and negative:

$$0 < y < \frac{1}{3} \quad y > 1$$

- (c) Draw a sketch of  $y$  on the direction field, paying particular attention to where  $y$  is increasing/decreasing and concave up/down. See the figure below.
- (d) Find an appropriate polynomial for  $F(y)$ .

SOLUTION: One example is

$$y' = -y(y-1)^2$$

10. Consider the DE:  $y dx + (2x - ye^y) dy = 0$ . Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of  $y$  alone (Hint: Find the integrating factor first).

SOLUTION: Let  $\mu$  be the integrating factor, and assume  $\mu$  is a function of  $y$  alone. Then

$$(\mu M)_y = \mu' \cdot M + \mu \cdot M_y = y\mu' + \mu$$

And

$$(\mu N)_x = 0 \cdot N + \mu \cdot 2$$

Setting these equal, we have the DE:

$$y\mu' = \mu \quad \Rightarrow \quad \int \frac{1}{\mu} d\mu = \int \frac{dy}{y}$$

or  $\mu = y$ . We can verify our answer, since  $y^2 dx + (2xy - y^2 e^y) dy = 0$  should now be exact.

11. Derive Euler's Method in stepping one unit in time. That is, given  $y' = f(t, y)$ , with  $y(t_0) = y_0$ , explain how we get  $y_1 = y(t_1)$ . Be sure to give an explanation, and not just write the formula down.

SOLUTION: We use the tangent line approximation to  $y(t)$  at  $t_0$ :

$$y - y_0 = y'(t_0)(t - t_0) \quad \Rightarrow \quad y = y_0 + f(t_0, y_0)(t - t_0)$$

then  $y_1$  is approximated by substituting  $t = t_1$ :

$$y_1 \approx y_0 + f(t_0, y_0)(t_1 - t_0)$$

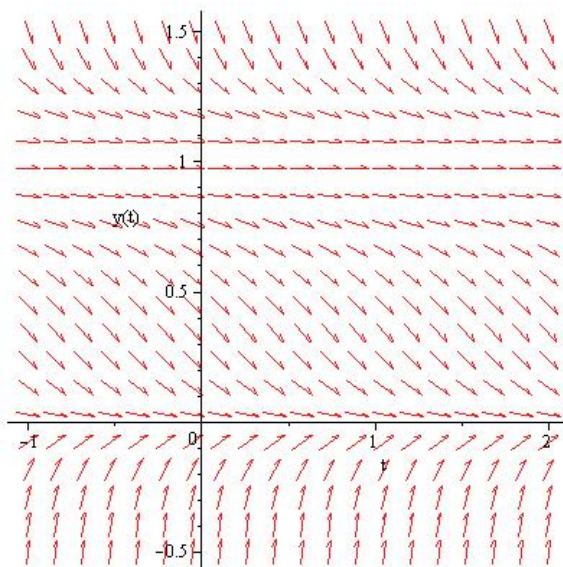


Figure 2: Figure for Exercise 9.

12. Use Euler's Method to determine  $y(1/2)$  if  $y' = 4 - t + 2y$ , with  $y(0) = 1$  and if we use one step in time. Repeat if we use 2 steps in time.

SOLUTION:

$$y'(0) = f(0, 1) = 4 - 0 + 2 = 6$$

so advancing  $1/2$  unit in time, we have:

$$y(1/2) \approx 1 + 6 \cdot (1/2) = 1 + 3 = 4$$

Taking two steps to do it, each step size would be  $1/4$ :

$$y(1/4) \approx 1 + 6 \cdot 1/4 = 5/2$$

Now,  $t = 1/4$  and  $y = 5/2$ , so the new slope is  $f(1/4, 5/2) = 35/4$ , and

$$y(1/2) \approx \frac{5}{2} + \frac{35}{4} \cdot \frac{1}{4} \approx 4.7$$

13. Suppose we have a vector field

$$\vec{F} = \langle 2x^2y + 2x, -(2xy^2 + 2y) \rangle$$

and we drop a particle into the  $(x, y)$  plane at the point  $(1, 2)$ . Find a curve along which the particle will travel by first constructing an appropriate differential equation and showing it is exact.

SOLUTION: First, the curve parametrized by  $\langle x(t), y(t) \rangle$  will satisfy the DE:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -(2xy^2 + 2y)/(2x^2 + 2x) \Rightarrow (2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

which is exact, and the general solution is

$$x^2y^2 + 2xy = k$$

Using the initial condition  $y(1) = 2$ , we get  $k = 8$ , or the solution is

$$x^2y^2 + 2xy = 8$$

14. Given the direction field below, find a differential equation that is consistent with it.

SOLUTION: Draw the corresponding figure in the  $(y, y')$  plane first. There we see that  $y = 0$  is unstable (make it a linear crossing), and  $y = 2$  is stable,  $y = 4$  is unstable. From the figure,

$$y' = y(y - 2)(y - 4)$$

will work.

15. Consider the direction field below, and answer the following questions:

(a) Is the DE possibly of the form  $y' = f(t)$ ?

SOLUTION: No. The isoclines would be vertical (consider, for example, a vertical line at  $t = -3$ ; the slopes are clearly not equal).

(b) Is the DE possible of the form  $y' = f(y)$ ?

SOLUTION: No. The isoclines would be horizontal (for example, look at a horizontal line at  $y = 1$ - Some slopes are zero, others are not).

(c) Is there an equilibrium solution? (If so, state it):

SOLUTION: Yes- At  $y = 0$ .

(d) Draw the solution corresponding to  $y(-1) = 1$ .

SOLUTION: Just draw a curve consistent with the arrows shown.

16. Evaluate the following integrals:

SOLUTIONS: You should be able to integrate by parts and use partial fractions fairly quickly at this point. Try checking your answers on Wolfram Alpha (for example, `integrate x^3exp(2x)` )

Some notes: The first integral should be done using a table, and the last should simplify a lot.

$$\int x^3 e^{2x} dx \quad \int \frac{x}{(x-1)(2-x)} dx \quad e^{-3} \int dt/t$$