Solutions to Review Questions: Exam 3

1. What are the assumptions we make for the solution y in

   - Chapter 6? SOLUTION: $y(t)$ is piecewise continuous and is of exponential order (so that $Y(s)$ exists).
   - Section 5.2? SOLUTION: $y(x)$ is analytic at $x = x_0$. That is,

   $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

   - Section 5.3-5.4: For the Euler Equations, $x^2 y'' + \alpha x y' + \beta y = 0$ SOLUTION: $y = x^r$

2. Finish the definitions:

   - The Heaviside function, $u_c(t)$:

   $$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \qquad c > 0$$

   - The Dirac $\delta-$function: $\delta(t - c)$

   $$\delta(t - c) = \lim_{\tau \to 0} d_\tau(t - c)$$

   where
   $$d_\tau(t - c) = \begin{cases} \frac{1}{2\tau} & \text{if } c - \tau < t < c + \tau \\ 0 & \text{elsewhere} \end{cases}$$

   - Define the convolution: $(f * g)(t)$

   $$(f * g)(t) = \int_0^t f(t - u) g(u)\, du$$

   - A function is of **exponential order** if:
   there are constants $M, k,$ and a so that

   $$|f(t)| \leq M e^{kt} \quad \text{for all} \quad t \geq a$$

   - A point $x = x_0$ is an ordinary point for $y'' + p(x)y' + q(x)y = g(x)$ if p, q, g are analytic at $x = x_0$. Practically speaking, if p, q, g are rational (a polynomial divided by a polynomial in lowest terms), then p, q, g will be analytic if the denominators are not zero at x_0.

3. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)

   $$f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 6 - t, & t \geq 2 \end{cases}$$

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,dt = \int_0^2 3\mathrm{e}^{-st}\,dt + \int_2^\infty (6-t)\mathrm{e}^{-st}\,dt$$

The second antiderivative is found by integration by parts:

$$\int_2^\infty (6-t)\mathrm{e}^{-st}\,dt \Rightarrow \begin{array}{ccc} \hline + & 6-t & \mathrm{e}^{-st} \\ - & -1 & (-1/s)\mathrm{e}^{-st} \\ + & 0 & (1/s^2)\mathrm{e}^{-st} \end{array} \Rightarrow \quad \mathrm{e}^{-st}\left(-\frac{6-t}{s} + \frac{1}{s^2}\right)\Big|_2^\infty$$

Putting it all together,

$$-\frac{3}{s}\mathrm{e}^{-st}\Big|_0^2 + \left(0 - \mathrm{e}^{-2s}\left(-\frac{4}{s}+\frac{1}{s^2}\right)\right) = -\frac{3\mathrm{e}^{-2s}}{s} + \frac{3}{s} + \frac{4\mathrm{e}^{-2s}}{s} - \frac{\mathrm{e}^{-2s}}{s^2} = \frac{3}{s} + \mathrm{e}^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

NOTE: Did you remember to *anti*differentiate in the third column?

(b)

$$f(t) = \begin{cases} \mathrm{e}^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,dt = \int_0^5 \mathrm{e}^{-st}\mathrm{e}^{-t}\,dt + \int_5^\infty -\mathrm{e}^{-st}\,dt = \int_0^5 \mathrm{e}^{-(s+1)t}\,dt + \int_5^\infty -\mathrm{e}^{-st}\,dt$$

Taking the antiderivatives,

$$-\frac{1}{s+1}\mathrm{e}^{-(s+1)t}\Big|_0^5 + \frac{1}{s}\mathrm{e}^{-st}\Big|_5^\infty = \frac{1}{s+1} - \frac{\mathrm{e}^{-5(s+1)}}{s+1} + 0 - \frac{\mathrm{e}^{-5s}}{s}$$

4. Check your answers to Problem 3 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

   (a) $f(t) = 3(u_0(t) - u_2(t)) + (6-t)u_2(t) = 3 - 3u_2(t) + (6-t)u_2(t) = 3 + (3-t)\,u_2(t)$
   For the second term, notice that the table entry is for $u_c(t)h(t-c)$. Therefore, if

   $$h(t-2) = 3-t \quad \text{then} \quad h(t) = 3-(t+2) = 1-t \quad \text{and} \quad H(s) = \frac{1}{s} - \frac{1}{s^2}$$

   Therefore, the overall transform is:

   $$\frac{3}{s} + \mathrm{e}^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

   (b) $f(t) = \mathrm{e}^{-t}\,(u_0(t) - u_5(t)) - u_5(t)$
   We can rewrite f in preparation for the transform:

   $$f(t) = \mathrm{e}^{-t}u_0(t) - \mathrm{e}^{-t}u_5(t) - u_5(t)$$

   For the middle term,

   $$h(t-5) = \mathrm{e}^{-t} \quad \Rightarrow \quad h(t) = \mathrm{e}^{-(t+5)} = \mathrm{e}^{-5}\mathrm{e}^{-t}$$

   so the overall transform is:

   $$F(s) = \frac{1}{s+1} - \mathrm{e}^{-5}\frac{\mathrm{e}^{-5s}}{s+1} - \frac{\mathrm{e}^{-5s}}{s}$$

5. Write the following functions in piecewise form (thus removing the Heaviside function):

   (a) $(t+2)u_2(t) + \sin(t)u_3(t) - (t+2)u_4(t)$
   SOLUTION: First, notice that $(t+2)$ is turned "on" at time 2. At time $t = 3$, $\sin(t)$ joins the first function, and at time $t = 4$, we subtract the function $t+2$ back off.
$$\begin{cases} 0 & \text{if } 0 \le t < 2 \\ t+2 & \text{if } 2 \le t < 3 \\ t+2+\sin(t) & \text{if } 3 \le t < 4 \\ \sin(t) & \text{if } t \ge 4 \end{cases}$$

   (b) $\displaystyle\sum_{n=1}^{4} u_{n\pi}(t)\sin(t - n\pi)$
   SOLUTION: First (you can determine this graphically) $\sin(t-\pi) = -\sin(t)$, and $\sin(t-2\pi) = \sin(t)$, and $\sin(t-3\pi) = -\sin(t)$, etc.- You should simplify these. Therefore:
$$\begin{cases} 0 & \text{if } 0 \le t < \pi \\ \sin(t-\pi) & \text{if } \pi \le t < 2\pi \\ \sin(t-\pi)+\sin(t-2\pi) & \text{if } 2\pi < t < 3\pi \\ \sin(t-\pi)+\sin(t-2\pi)+\sin(t-3\pi) & \text{if } 3\pi \le t < 4\pi \\ \sin(t-\pi)+\sin(t-2\pi)+\sin(t-3\pi)+\sin(t-4\pi) & \text{if } t \ge 4\pi \end{cases} =$$
$$\begin{cases} 0 & \text{if } 0 \le t < \pi \\ -\sin(t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi < t < 3\pi \\ -\sin(t) & \text{if } 3\pi \le t < 4\pi \\ 0 & \text{if } t \ge 4\pi \end{cases}$$

6. Is the given function of exponential order?

   (a) $\cos(t)$: SOLUTION: Since $\cos(t) \le 1$ for all t, $|\cos(t)| < e^{0t}$ for all t. (so $\cos(t)$ is of exponential order).

   (b) t^2: SOLUTION: $t^2 = e^{\ln(t^2)} = e^{2\ln(t)} < e^{2t}$ for $t > 0$, so t^2 is of exponential order.

   (c) e^{t^2}: Since $t^2 > at$ for any a and t large enough, we cannot bound e^{t^2} by e^{at}. Therefore, e^{t^2} is not of exponential order.

   (d) t^t: Similar to the last problem- For $t > 0$, $t^t = e^{t\ln(t)} \not\le e^{at}$ for any constant a and t large enough. Therefore, t^t is not of exponential order.

7. The tangent function is not piecewise continuous, because of the vertical asymptotes. It is also not of exponential order, because it goes to infinity over and over again in finite time (for example, at $t = \pi/2$, $t = 3\pi/2$, etc.).

8. Determine the Laplace transform:

   (a) t^2e^{-9t}
$$\frac{2}{(s+9)^3}$$

(b) $e^{2t} - t^3 - \sin(5t)$

$$\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$$

(c) $t^2 y'(t)$. Use Table Entry 16, $\mathcal{L}(-t^n f(t)) = F^{(n)}(s)$. In this case, $F(s) = sY(s) - y(0)$, so $F'(s) = sY'(s) + Y(s)$ and $F''(s) = sY''(s) + 2Y'(s)$.

(d) $e^{3t}\sin(4t)$

$$\frac{4}{(s-3)^2+16}$$

(e) $e^t\delta(t-3)$

In this case, notice that $f(t)\delta(t-c)$ is the same as $f(c)\delta(t-c)$, since the delta function is zero everywhere except at $t = c$. Therefore,

$$\mathcal{L}(e^t\delta(t-c)) = e^3 e^{-3s}$$

(f) $t^2 u_4(t)$

In this case, let $h(t-4) = t^2$, so that

$$h(t) = (t+4)^2 = t^2 + 8t + 16 \quad \Rightarrow \quad H(s) = \frac{2 + 8s + 16s^2}{s^3}$$

and the overall transform is $e^{-4s}H(s)$.

9. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$

$$\frac{2s-1}{s^2-4s+6} = \frac{2s-1}{(s^2-4s+4)+2} = 2\frac{s-1/2}{(s-2)^2+2} =$$

In the numerator, make $s - \frac{1}{2}$ into $s - 2 + \frac{3}{2}$, then

$$2\left(\frac{s-2}{(s-2)^2+2} + \frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{(s-2)^2+2}\right) \Rightarrow 2e^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}e^{2t}\sin(\sqrt{2}t)$$

(b) $\dfrac{7}{(s+3)^3} = \dfrac{7}{2!}\dfrac{2!}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2 e^{-3t}$

(c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)} = e^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{2}{s+2} \quad \Rightarrow \quad h(t) = 2e^t + 2e^{-2t}$$

and the overall inverse: $u_2(t)h(t-2)$.

(d) $\dfrac{3s-1}{2s^2-8s+14}$ Complete the square in the denominator, factoring the constants out:

$$\frac{3s-1}{2(s^2-8s+5)} = \frac{3}{2}\cdot\frac{s-1/3}{(s-2)^2+3} = \frac{3}{2}\left(\frac{s-2}{(s-2)^2+3} + \frac{5}{3}\cdot\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-2)^2+3}\right)$$

The inverse transform is:

$$\frac{3}{2}\mathrm{e}^{2t}\cos(\sqrt{3}t) + \frac{5}{2\sqrt{3}}\mathrm{e}^{2t}\sin(\sqrt{3}t)$$

(e) $\left(\mathrm{e}^{-2s}-\mathrm{e}^{-3s}\right)\dfrac{1}{s^2+s-6} = \left(\mathrm{e}^{-2s}-\mathrm{e}^{-3s}\right)H(s)$

Where:

$$H(s) = \frac{1}{s^2+s-6} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3}$$

so that

$$h(t) = \frac{1}{5}\mathrm{e}^{2t} - \frac{1}{5}\mathrm{e}^{-3t}$$

and the overall transform is:

$$u_2(t)h(t-2) - u_3(t)h(t-3)$$

10. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

(a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}$$

so that

$$Y(s) = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}$$

(b) $y'' + 9y = 10\mathrm{e}^{2t}$, $y(0) = -1$, $y'(0) = 5$

$$s^2Y + s - 5 + 9Y = \frac{10}{s-2} \Rightarrow Y(s) = \frac{10}{(s-2)(s^2+9)} - \frac{s-5}{s^2+9}$$

(c) $y'' - 4y' + 4y = t^2\mathrm{e}^t$, $y(0) = 0$, $y'(0) = 0$

$$(s^2-4s+4)Y = \frac{2}{(s-1)^3} \Rightarrow Y(s) = \frac{2}{(s-1)^3(s-2)^2}$$

11. Solve the given initial value problems using Laplace transforms:

(a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

$$Y = \frac{\mathrm{e}^{-5s}}{2s^2+s+2} = \mathrm{e}^{-5s}H(s)$$

where

$$H(s) = \frac{1}{2s^2+s+2} = \frac{1}{2}\frac{1}{s^2+\frac{1}{2}s+1} = \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

Therefore,

$$h(t) = \frac{2}{\sqrt{15}}\mathrm{e}^{-1/4\,t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

And the overall solution is $u_5(t)h(t-5)$

(b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

$$s^2Y + 3s - 10 + 6(sY+3) + 9Y = 0 \quad \Rightarrow \quad Y = -\frac{3s+8}{(s+3)^2}$$

Partial Fractions:

$$-\frac{3s+8}{(s+3)^2} = -\frac{3}{(s+3)} + \frac{1}{(s+3)^2} \Rightarrow y(t) = -3\mathrm{e}^{-3t} + t\mathrm{e}^{-3t}$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

$$Y = \mathrm{e}^{-s}\frac{1}{s(s-3)(s+1)} + \frac{1}{(s+1)(s-3)} = \mathrm{e}^{-s}H(s) + \frac{1}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1}$$

where

$$H(s) = \frac{1}{s(s-3)(s+1)} = -\frac{1}{3}\frac{1}{s} + \frac{1}{12}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}$$

so that

$$h(t) = -\frac{1}{3} + \frac{1}{12}\mathrm{e}^{3t} + \frac{1}{4}\mathrm{e}^{-t}$$

and the overall solution is:

$$y(t) = \frac{1}{4}\mathrm{e}^{3t} - \frac{1}{4}\mathrm{e}^{-t} + u_1(t)h(t-1)$$

(d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

$$Y = \mathrm{e}^{-\pi/2\,s}\frac{1}{s^2+4} + \frac{1}{s^2+4}$$

Therefore,

$$y(t) = \frac{1}{2}\sin(2t) + u_{\pi/2}(t)\frac{1}{2}\sin(2(t-\pi/2))$$

(e) $y'' + y = \sum_{k=1}^{\infty}\delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

$$Y(s) = \sum_{k=1}^{\infty}\mathrm{e}^{-2k\pi s}\frac{1}{s^2+1}$$

Therefore, term-by-term,

$$y(t) = \sum_{k=1}^{\infty} u_{2k\pi}(t)\sin(t - 2\pi k) = \sum_{k=1}^{\infty} u_{2\pi k}(t)\sin(t)$$

Piecewise,

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 2\pi \\ \sin(t) & \text{if} \quad 2\pi \le t < 4\pi \\ 2\sin(t) & \text{if} \quad 4\pi \le t < 6\pi \\ 3\sin(t) & \text{if} \quad 6\pi \le t < 8\pi \\ \vdots & \vdots \end{cases}$$

12. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

(a) $4y'' + 4y' + 17y = g(t) \qquad y(0) = 0, y'(0) = 0$

SOLUTION: First, note that

$$4s^2 + 4s + 17 = 4(s^2 + s + 17/4) = 4((s + 1/2)^2 + 4)$$

Therefore,

$$Y(s) = \frac{G(s)}{4s^2 + 4s + 17} = G(s) \cdot \frac{1}{8}\frac{2}{(s + \frac{1}{2})^2 + 2^2}$$

Therefore,

$$y(t) = g(t) * \frac{1}{8}e^{-t/2}\sin(2t)$$

(b) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t)$, with $y(0) = 1$ and $y'(0) = -1$.

SOLUTION: Take the Laplace transform of both sides:

$$(s^2Y - s + 1) + (sY - 1)_{\frac{5}{4}}Y = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

so that

$$Y(s) = \frac{1 - e^{-\pi s}}{s(s^2 + s + 5/4)} + \frac{s}{s^2 + s + 5/4}$$

For the second term,

$$\frac{s}{s^2 + s + 5/4} = \frac{s}{(s + \frac{1}{2})^2 + 1} = \frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} - \frac{1}{2}\frac{1}{(s + \frac{1}{2})^2 + 1}$$

For the first term, treat it like:

$$\left(e^{-0s} - e^{-\pi s}\right) H(s)$$

where

$$H(s) = \frac{1}{s} \cdot \frac{1}{s^2 + s + \frac{5}{4}} = \frac{1}{s} \cdot \frac{1}{(s + \frac{1}{2})^2 + 1}$$

so that

$$h(t) = 1 * \mathrm{e}^{-t/2}\sin(t)$$

Therefore, the overall answer is:

$$y(t) = h(t) - u_\pi(t)h(t-\pi) + \mathrm{e}^{-t/2}\left(\cos(t) - \frac{1}{2}\sin(t)\right)$$

13. Short Answer:

(a) $\displaystyle\int_0^\infty \sin(3t)\delta(t-\frac{\pi}{2})\,dt = \sin(3\pi/2) = -1$, since

$$\int_0^\infty f(t)\delta(t-c)\,dt = f(c)$$

(b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

$$y'(t) = \delta(t-c), \qquad y(0) = 0$$

SOLUTION:

$$sY = \mathrm{e}^{-cs} \Rightarrow Y = \frac{\mathrm{e}^{-cs}}{s}$$

so that $y(t) = u_c(t)$. Therefore, the "derivative" of the Heaviside function is the Dirac $\delta-$function!

(c) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2\int_0^t \cos(t-x)f(x)\,dx = \mathrm{e}^{-t}$$

(So only solve for the transform of $f(t)$, don't invert it back).

$$F(s) + 2F(s)\frac{s}{s^2+1} = \frac{1}{s+1} \quad \Rightarrow \quad F(s)\left(\frac{(s+1)^2}{s^2+1}\right) = \frac{1}{s+1}$$

so that

$$F(s) = \frac{s^2+1}{(s+1)^3}$$

(d) In order for the Laplace transform of f to exist, f must be?
f must be piecewise continuous and of exponential order

(e) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?
SOLUTION: No. Notice that the second derivative is not continuous at $x = 3$, but the second derivative of the power series would be.

(f) Use the table to find the Laplace transform of $\mathrm{e}^{-2t}\sinh(\sqrt{3}\,t)$.
SOLUTION: Use Table Entries 14 and 7:

$$\mathcal{L}\left(\mathrm{e}^{-2t}\sinh(\sqrt{3}t)\right) = F(s+2)$$

where $F(s)$ is the Laplace transform of $\sinh(\sqrt{3}t)$:

$$F(s) = \frac{\sqrt{3}}{s^2-9} \quad \Rightarrow \quad \text{Overall Answer:} \quad F(s+2) = \frac{\sqrt{3}}{(s+2)^2-9}$$

14. Let $f(t) = t$ and $g(t) = u_2(t)$.

   (a) Use the Laplace transform to compute $f * g$.
   To use the table,

$$\mathcal{L}(t * u_2(t)) = \frac{1}{s^2} \cdot \frac{\mathrm{e}^{-2s}}{s} = \mathrm{e}^{-2s}\frac{1}{s^3} = \mathrm{e}^{-2s}H(s)$$

   so that $h(t) = \frac{1}{2}t^2$. The inverse transform is then

$$u_2(t)\frac{1}{2}(t-2)^2$$

   (b) Verify your answer by directly computing the integral.
   By direct computation, we'll choose to "flip and shift" the function t:

$$f * g = \int_0^t (t-x)u_2(x)\,dx$$

   Notice that $u_2(x)$ is zero until $x = 2$, then $u_2(x) = 1$. Therefore, if $t \leq 2$, the integral is zero. If $t \geq 2$, then:

$$\int_0^t (t-x)u_2(x)\,dx = \int_2^t t - x\,dx = tx - \frac{1}{2}x^2\Big|_2^t = t^2 - \frac{1}{2}t^2 - 2t + 2 = \frac{1}{2}(t-2)^2$$

   valid for $t \geq 2$, zero before that. This means that the convolution is:

$$t * u_2(t) = \frac{1}{2}(t-2)^2 u_2(t)$$

15. If $a_0 = 1$, determine the coefficients a_n so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Try to identify the series represented by $\sum_{n=0}^{\infty} a_n x^n$.

SOLUTION: The recognition problem is a little difficult, but we should be able to get the coefficients:

$$\sum_{k=0}^{\infty} \left[(k+1)a_{k+1} + 2a_k\right] x^k = 0 \quad \Rightarrow \quad a_{k+1} = -\frac{2}{k+1}a_k$$

Just doing the straight computations, we get:

$$y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \cdots$$

To see the pattern, it is easiest to look at the general terms (Typically, I wouldn't ask for the recognition part on the exam, but you should be able to get the first few computations, as we did above):

$$\begin{array}{lll} a_1 & = -2a_0 & = \frac{(-2)}{1!}a_0 \\ a_2 & = -\frac{2}{2}a_0 = 2a_0 & = \frac{4}{2!}a_0 \\ a_3 & -\frac{2}{3}a_2 = -\frac{4}{3}a_0 & = \frac{-8}{3!}a_0 \\ \vdots & & \vdots \end{array}$$

The series is for $\mathrm{e}^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}$

16. Write the following as a single sum in the form $\sum_{k=2}^{\infty} c_k(x-1)^k$ (with a few terms in the front):

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + x(x-2)\sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In front of the second sum we have $x^2 - 2x$, but we can't bring that directly into the sum since we have powers of $(x-1)$. But, we might recognize that:

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x-1)^2 - 1$$

Therefore, the second sum can be expanded into two sums:

$$((x-1)^2 - 1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = (x-1)^2\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} =$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

Now we have three sums to work with

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In the first sum, the first non-zero term has $(x-1)^0$, the second sum begins with $(x-1)^2$, and the last sum starts with $(x-1)^0$. We could shift the second index to start at $n = 0$, but then the sum begins with $(x-1)^1$. We'll have to break off the constant terms from the first two sums:

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1)(x-1)^{n-2}$$

and similarly

$$-\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = -a_1 - \sum_{n=2}^{\infty} na_n(x-1)^{n-1}$$

Now we can bring all three sums together. In the first sum, we'll substitute $k = n-2$ (or $n = k+2$). In the middle sum, $k = n+1$ (or $n = k-1$), and in the third sum, $k = n-1$ (or $n = k+1$). With these substitutions, we get:

$$2a_2 - a_0 + \sum_{k=1}^{\infty} \left((k+2)(k+1)a_{k+2} + (k-1)a_{k-1} - (k+1)a_{k+1}\right)(x-1)^k$$

NOTE: The question asked for the index to start at $k = 2$ instead of $k = 1$- It's OK to do it either way; mainly, I wanted to see you put the sums together as one.

17. Characterize ALL (continuous or not) solutions to

$$y'' + 4y = u_1(t), \qquad y(0) = 1, y'(0) = 1$$

SOLUTION: The idea behind this question is to get you to think about the kinds of solutions we get from the Laplace transform. If we do not require y to be continuous, then this DE is actually two differential equations:

$$y'' + 4y = 0, \qquad y(0) = 1, y'(0) = 1 \qquad \text{valid for } t \le 1$$

And

$$y'' + 4y = 1 \qquad y(1), y'(1) \text{ arbitrary }, \text{ valid for } t > 1$$

The general solution is then:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \le 1 \\ c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

If we require $y(t)$ to be continuous (a very common assumption), then we get the answer that comes from using Laplace transforms. Writing the answer in piecewise form:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \le 1 \\ -\frac{1}{4}\cos(2(t-1)) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

18. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to convert the following DE into a linear first order DE in $Y(s)$ (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

SOLUTION: For the first part, use Table Entry 19. In particular,

$$\mathcal{L}(tf(t)) = -F'(s)$$

where, in our case, $f(t) = y'(t)$, so that $F(s) = sY - y(0)$. Therefore,

$$\mathcal{L}(tf(t)) = -(Y - sY') = sY' - Y$$

Substituting this into the DE, we get:

$$Y' + \left(\frac{s^2 - 3s - 6}{3s}\right)Y = \frac{1}{s}$$

19. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

Substituting our power series in for y, y', y'':

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Noting that in the second sum we could start at $n = 0$, since the first term (constant term) would be zero anyway, we can start all series with a constant term:

$$\sum_{k=0}^{\infty} (2(k+2)(k+1)a_{k+2} + ka_k + 3a_k)\, x^k = 0$$

From which we get the recurrence relation:

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)}\, a_k$$

(b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$
Substituting our power series in for y, y', y'':

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

The two middle sums can have their respective index taken down by one (so that formally the series would start with $0x^0$):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now make all the indices the same. To do this, in the first sum make $k = n-2$, in the second sum take $k = n-1$. Doing this and collecting terms:

$$\sum_{k=0}^{\infty} ((k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + (k-1)a_k)\, x^k = 0$$

So we get the recursion:

$$a_{k+2} = \frac{(k+1)k\, a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

(c) $y'' - xy' - y = 0$, $x_0 = 1$
Done in class;

$$a_{n+2} = \frac{1}{n+2}\,(a_{n+1} + a_n)$$

20. Exercises with the table:

(a) Use table entries 5 and 14 to prove the formula for 9.

SOLUTION: Prove formula #9 using 5 and 14:

$$\mathcal{L}(\mathrm{e}^{at}\sin(bt)) = F(s-a)$$

where

$$F(s) = \mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2} \quad \Rightarrow \quad \frac{b}{(s-a)^2+b^2}$$

Therefore,

$$\mathcal{L}(\mathrm{e}^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

(b) Show that you can use table entry 19 to find the Laplace transform of $t^2\delta(t-3)$ (verify your answer using a property of the δ function).

SOLUTION: Using Entry 19, the Laplace transform of $t^2\delta(t-3)$ is the second derivative of the Laplace transform of $\delta(t-3)$. That is, using

$$F(s) = \mathrm{e}^{-3s}$$

then

$$\mathcal{L}(t^2\delta(t-3)) = F''(s) = 9\mathrm{e}^{-3s}$$

And this is the same as:

$$\int_0^\infty \mathrm{e}^{-st}t^2\delta(t-3)\,dt = 9\mathrm{e}^{-3s}$$

(c) Prove (using the definition of $\mathcal{L}$) table entries 12 and 13.

SOLUTION: 12 is a special case of 13, so we prove 13 using the definition:

$$\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty \mathrm{e}^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty \mathrm{e}^{-st}f(t-c)\,dt$$

We want this answer to be the following (with a different variable of integration):

$$\mathrm{e}^{-cs}F(s) = \mathrm{e}^{-cs}\int_0^\infty \mathrm{e}^{-sw}f(w)\,dw = \int_0^\infty \mathrm{e}^{-s(w+c)}f(w)\,dw$$

We can connect the two by taking $w = t-c$ (so that $t = w+c$), and then (remember to change the bounds!):

$$\int_c^\infty \mathrm{e}^{-st}f(t-c)\,dt = \int_0^\infty \mathrm{e}^{-s(w+c)}f(w)\,dw$$

And we're done.

(d) Prove (using the definition of $\mathcal{L}$) a formula (similar to 18) for $\mathcal{L}(y'''(t))$.

SOLUTION: I wanted you to recall how we got those definitions in the past (integrating by parts):

$$\mathcal{L}(y'''(t)) = \int_0^\infty \mathrm{e}^{-st}y'''(t)\,dt$$

Integration by parts using a table:

$$\begin{array}{crr} + & \mathrm{e}^{-st} & y'''(t) \\ - & -s\mathrm{e}^{-st} & y''(t) \\ + & s^2\mathrm{e}^{-st} & y'(t) \\ - & -s^3\mathrm{e}^{-st} & y(t) \end{array} \quad \Rightarrow \quad \left(\mathrm{e}^{-st}\left(y''(t)+sy'(t)+s^2y(t)\right)\right|_{t=0}^{\infty} + s^3\int_0^{\infty} \mathrm{e}^{-st}y(t)\,dt$$

At infinity, these terms all go to zero (otherwise, the Laplace transform wouldn't exist), so we get:

$$s^3 - (y''(0) + sy'(0) + s^2y(0)) = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

21. Find the first 5 terms of the power series solution to $\mathrm{e}^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

Compute the derivatives directly, then (don't forget to divide by $n!$):

$$y(x) = 1 - x - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \ldots$$

22. Find the radius of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}x^n$

SOLUTION:

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{n}}|x| = |x|$$

So by the ratio test, the series will converge (absolutely) if $|x| < 1$ (so the radius is 1).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$

SOLUTION: Simplifying the limit in the ratio test, we get

$$\lim_{n\to\infty} 2\sqrt{\frac{n}{n+1}}|x+3| = 2|x+3|$$

Therefore, by the ratio test, the series will converge absolutely if $2|x+3| < 1$, or if $|x+3| < 1/2$ (and this is our radius). For the interval of convergence, we have to check the points $x = -7/2$ and $x = -5/2$ separately. For $x = -7/2$, the series diverges ($p-$test), and for $x = -5/2$, the series converges by the alternating series test.

NOTE: If you don't recall those tests, you probably ought to review them, but I won't make you recall them for the exam this week.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{n^n}$ (A little tricky)

SOLUTION: This one is a little tricky because we need to recall the definition of e:

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

In this case, using the Ratio Test, we have:

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} |x| = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)(n+1)^n} |x| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x| = \frac{|x|}{e}$$

so the radius of convergence is e.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$

SOLUTION: This is definitely similar to problems on exams/quizzes. The Ratio Test simplifies to:

$$\frac{1}{5} \lim_{n\to\infty} \frac{n}{n+1} |3x-2| = \frac{|3x-2|}{5}$$

To converge absolutely, $|3x-2| < 5$. To get the radius of convergence, we need to have the form $|x-a| < \rho$, so in this case, we simplify to get:

$$3\left|x - \frac{2}{3}\right| < 5 \quad \Rightarrow \quad \left|x - \frac{2}{3}\right| < \frac{5}{3}$$

Now we have to check the endpoints separately, which are $x = -1$ and $x = 7/3$:

- At $x = -1$, the sum becomes:

  $$\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

  This is an alternating harmonic series, which converges (but not absolutely).
- At $x = 7/3$, the sum becomes a harmonic series, which diverges.

The interval of convergence is: $[-1, \frac{7}{3})$