

# Math 244

## Homework To Replace 7.1/7.2

### Modeling

1. Exercise 22 (Section 7.1, p 363)

### Conversions back and forth

2. Short Answer:
  - (a) Is every second order linear homogeneous differential equation (with constant coefficients) equivalent to a system of first order equations?
  - (b) Can every  $2 \times 2$  system of DEs be converted into an equivalent second order system? (Hint: To do our technique, what must be true?)
3. Solve the system of equations given by first converting it into a second order linear ODE (then use Chapter 3 methods):

$$(a) \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} \qquad (b) \mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$$

4. Convert the following second order differential equations into a system of autonomous, first order equations. Using methods from Chapter 3, give the solution to the system. An example follows before the exercises:

$$y'' + 3y' + 2y = 0$$

SOLUTION: We'll get the homogenous solution first. The roots to the characteristic equation are  $-1, -2$ . The general solution is:

$$y = C_1 e^{-t} + C_2 e^{-2t}$$

To get an equivalent system, let  $x_1 = y$  and  $x_2 = y'$ . Then

$$x_1' = y' = x_2 \qquad x_2' = y'' = -2y - 3y' = -2x_1 - 3x_2$$

so the system is (in matrix-vector form):

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}$$

Since  $x_1 = y$ , then  $x_1 = C_1 e^{-t} + C_2 e^{-2t}$ . Since  $x_2 = y'$ , then  $x_2 = -C_1 e^{-t} - 2C_2 e^{-2t}$ . In vector form, this means our solution is:

$$\mathbf{x} = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Here we go:

- |                          |                         |
|--------------------------|-------------------------|
| (a) $y'' + 4y' + 3y = 0$ | (c) $y'' + 4y = 0$      |
| (b) $y'' + 5y' = 0$      | (d) $y'' - 2y' + y = 0$ |

## Parametric Functions

5. If  $\mathbf{x}$  is as defined below, compute  $\mathbf{x}'(t)$ , and  $\int_0^1 \mathbf{x}(t) dt$ :

$$\mathbf{x}(t) = \begin{bmatrix} t^2 - 3 \\ 3e^t - 2e^{3t} \end{bmatrix}$$

6. If  $\mathbf{y}(t) = A(t)\mathbf{c}$  is as defined below, compute  $\mathbf{y}'(t)$ , and  $\int_0^1 \mathbf{y}(t) dt$ . Are these the same as  $A'(t)\mathbf{c}$  and  $\int A(t) dt \mathbf{c}$ ?

$$\mathbf{y}(t) = \begin{bmatrix} t & 2t \\ 1 & \sin(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

## Systems of Equations

7. Give the solution to each system. If it has an infinite number of solutions, give your answer in vector form:

$$\begin{array}{rcl} 3x + 2y & = & 1 \\ 2x - y & = & 3 \end{array} \quad \begin{array}{rcl} 3x + 2y & = & 1 \\ 6x + 4y & = & 3 \end{array} \quad \begin{array}{rcl} 3x + 2y & = & 1 \\ 6x + 4y & = & 2 \end{array}$$

8. Write each of the previous systems in matrix-vector form. Verify that the determinant of the first matrix is not zero, but is zero for the second and third.
9. Write each system of differential equations in matrix-vector form or write the system from the matrix-vector form:

$$\begin{array}{l} x_1' = 3x_1 - x_2 \\ x_2' = 9x_1 - 3x_2 \end{array} \quad \mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

10. Find the equilibrium solutions to the previous autonomous linear differential equations:

## Matrix Algebra and Vectors

11. Exercise 1, 4 (Section 7.2, p. 371-373)

For Exercises 3-8 below, define

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and  $I$  is the identity matrix:

12. Compute  $(B - 4I)\mathbf{b}$
13. Compute  $\det(B - 4I)$
14. Are  $AB$  and  $BA$  the same?
15. Compute  $A^{-1}\mathbf{b}$
16. Verify:  $A\mathbf{b} - 3\mathbf{b} = (A - 3I)\mathbf{b}$
17. Compute  $B^T B$ ,  $\text{Tr}(A)$ , and  $\text{Tr}(B)$
18. Exercises 22, 23 (Section 7.2, p. 373)