Selected Solutions, Section 5.3

1. We determine the derivatives by simply differentiating and evaluating at the given point. We will go ahead and use $y(x)$ in place of $\phi(x)$. Technically speaking, these are not the same thing (ϕ is the series approximation to the true solution y):

$$y(0) = 1 \qquad y'(0) = 0$$

$$y'' = -xy' - y \quad \Rightarrow y''(0) = -1$$
$$y''' = -y' - xy'' - y' = -2y' - xy'' \Rightarrow y'''(0) = 0$$
$$y^{iv} = -2y'' - y'' - xy''' = -3y'' - xy''' \Rightarrow y^{iv}(0) = 3$$

3. Similar to (1):

$$y(1) = 2 \qquad y'(1) = 0$$
$$x^2y'' + (1 + x)y' + 3\ln(x)y = 0 \Rightarrow y'' + 2(0) + 3(0)(2) = 0 \Rightarrow y'' = 0$$

Probably best to just differentiate in place, simplify, then evaluate at $x = 1$:

$$2xy'' + x^2y''' + y' + (1 + x)y'' + \frac{3}{x}y + 3\ln(x)y' = 0 \Rightarrow$$

$$x^2y''' + (1 + 3x)y'' + (1 + 3\ln(x))y' + \frac{3y}{x} = 0 \Rightarrow$$
$$y''' + 4(0) + 0 + 6 = 0 \Rightarrow y'''(1) = -6$$

Similarly, for the fourth derivative:

$$2xy''' + x^2y^{iv} + 3y'' + (1 + 3x)y''' + \frac{3}{x}y' + (1 + 3\ln(x))y'' - \frac{3}{x^2}y + \frac{3}{x}y' = 0$$

so that $y^{iv}(1) = 42$.

5. In this case, $p(x) = 4$ and $q(x) = 6x$. These functions have a "radius of convergence" of ∞ (since they are finite sums), so we expect that, no matter the base point, the solution will also have a radius of convergence of ∞.

6. In this case,

$$p(x) = \frac{x}{x^2 - 2x - 3} = \frac{x}{(x-3)(x+1)} \qquad q(x) = \frac{4}{(x-3)(x+1)}$$

Both p and q fail to exist at $x = 3$ and $x = -1$. Therefore, we expect that these points will not be included in the interval of convergence for $y(x)$. It is perhaps easiest to construct a number line:

$x_0 = -4$ $\qquad$ -1 $\qquad$ $x_0 = 0$ $\qquad$ 3 $\qquad$ $x_0 = 4$

- For $x_0 = -4$, the closest "bad point" is -1, which is 3 units away. Therefore, in this case we expect the radius of convergence to be 3.
- For $x_0 = 0$, the closest bad point is -1, which is 1 unit away. The radius of convergence would be 1.
- For $x_0 = 4$, the closest bad point is 3, which is again 1 unit away, so the radius of convergence is 1.

10. The Chebyshev Equation comes up in some applications. In place of the instructions, **just find the recurrence relation**.

We assume the usual ansatz, and substitute into the DE to get:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

The first sum should be split into two so that we have 4 sums total:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

In order for all sums to begin with x^0, we could simply start the middle two sums at $n = 0$, since we would just be adding "0" to the sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

Substitute $k = n - 2$ into the first sum, the rest take $n - k$ to get:

$$\sum_{k=0}^{\infty} \Big((k+2)(k+1)a_{k+2} + (\alpha^2 - k^2)a_k\Big)\, x^k = 0$$

From which we get the recurrence relation (for $k = 0, 1, 2, 3, \cdots$)

$$a_{k+2} = \frac{k^2 - \alpha^2}{(k+2)(k+1)} a_k$$

(and that's as far as you need to take this one).

Problems 11 and 12 are very similar to Problems 1 and 3. The series is found by differentiating and substituting in the particular values of x. Notice that, to get the linearly independent functions, first set $y(0) = 1$, $y'(0) = 0$, then $y(0) = 0$ and $y'(0) = 1$ (this is equivalent to setting $a_0 = 1, a_1 = 0$, then taking $a_0 = 0, a_1 = 1$ from the previous section).

11. For the radius of convergence, we expect ∞, since $p(x) = 0$ and $q(x) = \sin(x)$ are analytic for all x.

We differentiate a few times, then evaluate at $x = 0$:

$$y'' = -\sin(x)y$$

$$y''' = -\cos(x)y - \sin(x)y'$$
$$y^{iv} = \sin(x)y - 2\cos(x)y' - \sin(x)y''$$
$$y^{v} = \cos(x)y + 3\sin(x)y' - 3\cos(x)y'' - \sin(x)y'''$$
$$y^{vi} = -\sin(x)y + 4\cos(x)y' + 6\sin(x)y'' - 4\cos(x)y''' - \sin(x)y^{iv}$$
$$y^{vii} = -\cos(x)y - 5\cos(x)y' + 10\sin(x)y'' + 10\cos(x)y''' - 5\cos(x)y^{iv} - \sin(x)y^{v}$$

Cool! Do you see Pascal's Triangle?

We'll simplify (a lot) by taking $x = 0$:

	$y(0) = 1, y'(0) = 0$	$y(0) = 0, y'(0) = 1$
$y'' = 0$	0	0
$y''' = -y$	-1	0
$y^{(4)} = -2y'$	0	-2
$y^{(5)} = y - 3y'' = y$	1	0
$y^{(6)} = 4y' - 4y'''$	4	4
$y^{(7)} = -y + 10y'' - 5y^{(4)} = -y - 5y^{(4)}$	-1	10

so:

$$y_1(x) = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{4}{6!}x^6 - \frac{1}{7!}x^7 + \dots$$
$$y_2(x) = x - \frac{2}{4!}x^4 + \frac{4}{6!}x^6 + \frac{10}{7!}x^7 + \dots$$

12. In this case, we also expect the radius of convergence to be infinite, since $q(x) = x\mathrm{e}^{-x}$ is analytic for all x.

As in problem 11, differentiate and evaluate the derivatives at the origin:

$$\mathrm{e}^x y'' + xy = 0$$

$$\mathrm{e}^x y''' + \mathrm{e}^x y'' + xy' + y = 0$$
$$\mathrm{e}^x y^{(4)} + 2\mathrm{e}^x y''' + \mathrm{e}^x y'' + xy'' + 2y' = 0$$
$$\mathrm{e}^x y^{(5)} + 3\mathrm{e}^x y^{(4)} + 3\mathrm{e}^x y''' + \mathrm{e}^x y'' + xy''' + 3y'' = 0$$

Evaluating at $x = 0$,

	$y(0) = 1, y'(0) = 0$	$y(0) = 0, y'(0) = 1$
$y'' = 0$	0	0
$y''' = -y'' - y$	-1	0
$y^{(4)} = -2y''' - y'' - 2y'$	2	-2
$y^{(5)} = -3y^{(4)} - 3y''' - 4y''$	-3	6

so that:

$$y_1(x) = 1 - \frac{1}{3!}x^3 + \frac{2}{4!}x^4 - \frac{3}{5!}x^5 + \dots$$

$$y_2(x) = x - \frac{2}{4!}x^4 + \frac{6}{5!}x^5 + \dots$$

15. Using a different argument from the text, if x, x^2 are solutions to the differential equation, we can substitute them in and see what we get.

For $y = x$, $P(x)y'' + Q(x)y' + R(x)y = 0$ becomes:

$$Q(x) = -xR(x)$$

For $y = x^2$, the differential equation becomes:

$$2P(x) + 2xQ(x) + x^2R(x) = 0 \quad \Rightarrow \quad 2P + 2x(-xR) + x^2R = 0 \quad \Rightarrow \quad P = -\frac{x^2}{2}R$$

Substituting these in,

$$p(x) = \frac{Q(x)}{P(x)} = \frac{2xR(x)}{x^2R(x)} = \frac{2}{x}$$

and

$$q(x) = \frac{R(x)}{P(x)} = \frac{-2R(x)}{x^2R(x)} = -\frac{2}{x^2}$$

so that the point $x = 0$ is a singular point (not an ordinary point).

16-18 These are kind of fun- And a little simpler, since they are only first order. Number 18 is more difficult and would require us multiplying the Maclaurin series for e^{x^2} and the series for $y(x)$ together (that's why they only ask for three terms)- If we only need three terms, use the method from the first part of this set (computing $y'(0), y''(0), y'''(0)$, etc).

For example, in 16:

$$y' - y = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n = 0$$

They both already start with x^0 power. Let $m = n - 1$ in the first sum, and $m = n$ in the second to get

$$\sum_{m=0}^{\infty} ((m+1)a_{m+1} - a_m)x^m = 0$$

The recurrence relation is then the following, where a_0 is given (it is the initial condition).

$$a_{m+1} = \frac{a_m}{m+1} \qquad \text{for } m = 0, 1, 2, \ldots$$

so for example,

$$a_1 = a_0, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0 \qquad a_3 = \frac{1}{3}a_2 = \frac{1}{3 \cdot 2}a_0 \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4 \cdot 3 \cdot 2}a_0$$

so apparently

$$a_n = \frac{a_0}{n!} \qquad \Rightarrow \qquad y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

which we recognize as $y(x) = a_0 \mathrm{e}^x$.