Appendix B solutions

There are two techniques- One we discussed in class, that simply compute the derivatives, $y^{(n)}(t_0)$ and the text, which writes out the polynomial in terms of the coefficients a_n. Recall that, from the Taylor series, we have the relationship:

$$a_n = \frac{y^{(n)}(t_0)}{n!}$$

Typically, $t_0 = 0$, and these are the coefficients for the Maclaurin series of y.

1. $y' = y$

   - Using the textbook technique: $y(t) = a_0 + a_1 t + \cdots$, $y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots$:

     $$a_1 + 2a_2 t + 3a_3 t^2 + \cdots = a_0 + a_1 t + \cdots$$

     These two polynomials are equal for all time, meaning that the coefficients are the same. We can write all of the coefficients in terms of a_0 (which is $y(0)$):

     $$\begin{array}{lllr} a_1 & = a_0 & & \\ 2a_2 & = a_1 & = a_0 & a_2 = \frac{1}{2}a_0 \\ 3a_3 & = a_2 & & a_3 = \frac{1}{2\cdot 3}a_0 \\ 4a_4 & = a_3 & & a_4 = \frac{1}{4!}a_0 \end{array}$$

     And we see that $a_n = \frac{1}{n!}a_0$. Therefore, the solution is:

     $$y(t) = a_0\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots\right)$$

     You might recall that this is actually the series for the exponential function: $y(t) = a_0 e^t$, but that wasn't asked.

   - Using our alternative method: $y(0) = a_0$, then $y'(0) = y(0) = a_0$, and $y''(0) = y'(0) = y(0) = a_0$, and so all of the derivatives are a_0. Remember that, to get a_n, we divide by $n!$, and we get the same series as the previous answer.

2. $y' = -y + 1$

   - Using the textbook technique, and grouping the constant terms together on the right side:

     $$y' = -y + 1 \quad \Rightarrow \quad a_1 + 2a_2 t + 3a_3 t^2 + \cdots = (1 - a_0) - a_1 t - a_2 t^2 - \cdots$$

     Equating coefficients and writing things in terms of a_0, we have:

     $$\begin{array}{llr} a_1 & = 1 - a_0 & \\ 2a_2 & = -a_1 & a_2 = \frac{1}{2}(a_0 - 1) \\ 3a_3 & = -a_2 & a_3 = \frac{-1}{2\cdot 3}(a_0 - 1) \\ 4a_4 & = -a_3 & a_4 = \frac{1}{4!}(a_0 - 1) \end{array}$$

- Using our derivating technique:

$$\begin{array}{ll} y(0) & = a_0 \\ y' & = -y+1 \quad \Rightarrow y'(0) = 1-a_0 \\ y'' & = -y' \quad \Rightarrow y''(0) = -(1-a_0) \\ y''' & = -y'' \quad \Rightarrow y'''(0) = (1-a_0) \end{array}$$

In either case, we see that $(1-a_0)$ can be factored out of everything except the first term:

$$y(t) = a_0 + (1-a_0)\left(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - \frac{1}{4!}t^4 + \cdots\right)$$

3. In this case, we'll just do one iteration, then give the solution:

$$\begin{array}{rcl} y' = -2ty & \Rightarrow & y'(0) = 0 \\ y'' = -2y - 2ty' & \Rightarrow & y''(0) = -2y(0) = -2a_0 \\ y''' = -4y' - 2ty'' & \Rightarrow & y'''(0) = 0 \\ y^{(4)} = -6y'' - 2ty''' & \Rightarrow & y^{(4)}(0) = 12a_0 \end{array}$$

We should see that

$$y(t) = a_0 - \frac{2a_0}{2}t^2 + \frac{12}{24}t^4 + \cdots = a_0\left(1 - t^2 + \frac{1}{2}t^4 + \cdots\right)$$

4. Same kind of thing as (3) for (4) - Remember to use the product rule. In this case, the textbook technique would be:

$$a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \cdots = 1 + a_0t^2 + a_2t^3 + a_2t^4 + \cdots$$

Therefore,

$$a_1 = 1 \quad a_2 = 0 \quad a_3 = a_0/3 \quad a_4 = 1/4 \quad a_5 = 0\cdots$$

Therefore, we get:

$$y(t) = a_0 + t + \frac{a_0}{3}t^3 + \frac{1}{4}t^4 + \cdots$$

5. For problems 5-8, it's probably easiest to use the derivative technique (rather than the series comparison technique from the book). For 5, we should get:

$$\begin{array}{rcl} y(0) & = a_0 & \\ y' = -y + \mathrm{e}^{2t} & \Rightarrow & y'(0) = 1 - a_0 \\ y'' = -y' + 2\mathrm{e}^{2t} & \Rightarrow & y''(0) = -1 + a_0 + 2 = 1 + a_0 \end{array}$$

and so on, leading to:

$$y(t) = a_0 + (1-a_0)t + \frac{1+a_0}{2}t^2 + \frac{3-a_0}{3!}t^3 + \frac{5+a_0}{4!}t^4 + \cdots$$

6. Same technique as 5, the series is:

$$y(t) = a_0 + 2a_0 t + \frac{4a_0 + 1}{2} t^2 + \frac{4a_0 + 1}{3} t^3 + \frac{16a_0 + 3}{24} t^4 + \cdots$$

7. This is a second order DE, so $y(0) = a_0$, $y'(0) = a_1$, and the remaining coefficients will be in terms of a_0, a_1:

$$\begin{aligned} y'' &= -2y + \cos(t) & \Rightarrow \quad & y''(0) = -2a_0 + 1 \\ y''' &= -2y' - \sin(t) & \Rightarrow \quad & y'''(0) = -2a_1 \\ y^{(4)} &= -2y'' - \cos(t) & \Rightarrow \quad & y^{(4)}(0) = 4a_0 - 3 \end{aligned}$$

which gives:

$$y(t) = a_0 + a_1 t + \frac{-2a_0 + 1}{2} t^2 - \frac{2a_1}{3!} t^3 + \frac{4a_0 - 3}{4!} t^4 + \cdots$$

8. Very similar to the previous problem:

$$y(t) = a_0 + a_1 t - \frac{a_0 + 5a_1}{2!} t^2 + \frac{4 + 5a_0 + 24a_1}{3!} t^3 - \frac{20 + 24a_0 + 115a_1}{4!} t^4 + \cdots$$