# General Review SOLUTIONS

(Corrected May 13, 2004: Prob 15(e), 17(b))

1. Solve (use any method if not otherwise specified):

   (a) $(2x-3x^2)\frac{dx}{dt} = t\cos(t)$ This is separable, and separated.

   $$\int 2x - 3x^2\, dx = \int t\cos(t)\, dt \Rightarrow x^2 - x^3 = \cos(t) + t\sin(t) + C$$

   You can leave your answer in implicit form.

   (b) $y'' + 2y' + y = \sin(3x)$ Get the homog part then the particular solution (or use Laplace):

   $$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1 \Rightarrow y_h(t) = \mathrm{e}^{-t}\,(C_1 + C_2 t)$$

   For the particular solution, (Undet Coefs), $y_p = A\cos(3x) + B\sin(3x)$. Substitute to get:

   $$(A - 6B - 9A)\sin(3x) + (B + 6A - 9B)\cos(3x) = \sin(3x)$$

   so that $A = -2/25$, $B = -3/50$. Altogether,

   $$y(t) = \mathrm{e}^{-t}\,(C_1 + C_2 t) - \frac{2}{25}\sin(3x) - \frac{3}{50}\cos(3x)$$

   This one is a little messy with Laplace transforms.

   (c) $y'' - 3y' + 2y = \mathrm{e}^{2t}$ Same technique here. The roots to the characteristic equation are $r = 1, 2$, so the homogeneous part of the solution is:

   $$y_h(t) = C_1\mathrm{e}^{t} + C_2\mathrm{e}^{2t}$$

   Initially, we guess that $y_p(t) = A\mathrm{e}^{2t}$, but that is part of $y_h$, so multiply by $t$: $y_p = At\mathrm{e}^{2t}$. Now substitute into the D.E. to get: $A = 1$. The full solution is

   $$y(t) = C_1\mathrm{e}^{t} + C_2\mathrm{e}^{2t} + t\mathrm{e}^{2t}$$

   (d) $x' = \sqrt{t}\mathrm{e}^{-t} - x$. This is a linear differential equation, with integrating factor; $x' + x = \sqrt{t}\mathrm{e}^{-t}$. The integrating factor is $\mathrm{e}^{\int 1\, dt} = \mathrm{e}^{t}$. Therefore,

   $$(x\mathrm{e}^{t})' = \sqrt{t} \Rightarrow x\mathrm{e}^{t} = \frac{2}{3}t^{3/2} + C \Rightarrow x = \left(\frac{2}{3}t^{3/2} + C\right)\mathrm{e}^{-t}$$

   (e) $x' = 2 + 2t^2 + x + t^2 x$. This is a linear differential equation: $x' - (1+t^2)x = 2(1+t^2)$. The integrating factor is:

   $$\mathrm{e}^{\int -(1+t^2)\, dt} = \mathrm{e}^{-t-(1/3)t^3}$$

   So we solve the following (to integrate, let $u = t + (1/3)t^2$)

   $$\left(x\mathrm{e}^{-t-(1/3)t^3}\right)' = 2(1+t^2)\mathrm{e}^{-t-(1/3)t^3} \Rightarrow \left(x\mathrm{e}^{-t-(1/3)t^3}\right) = -2\mathrm{e}^{-t-(1/3)t^3} + C \Rightarrow x = -2 + C\mathrm{e}^{t+(1/3)t^3}$$

2. Obtain the general solution in terms of $\alpha$, then determine a value of $\alpha$ so that $y(t) \to 0$ as $t \to \infty$:

   $$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

   The general solution (before initial conditions):

   $$y(t) = C_1\mathrm{e}^{3t} + C_2\mathrm{e}^{-2t}$$

   With the initial conditions,

   $$1 = C_1 + C_2 \quad \alpha = 3C_1 - 2C_2 \Rightarrow C_1 = \frac{2+\alpha}{5}, \quad C_2 = \frac{3-\alpha}{5}$$

   Therefore,

   $$y(t) = \frac{2+\alpha}{5}\mathrm{e}^{3t} + \frac{3-\alpha}{5}\mathrm{e}^{-2t}$$

   For $y(t) \to 0$, we must have $\alpha = -2$ (to zero out the first term).

3. The Wronskian of two functions is $W(t) = t^2 - 4$. Are they two linearly independent solutions to a second order linear differential equation? Why or why not? If the interval for the solution does not include $\pm 2$, then yes, these could be two linearly independent solutions to a second order linear differential equation.

4. Compute $\mathcal{L}(\cos(t))$ by using the definition of the Laplace transform and integration by parts twice:

$$\int_0^\infty e^{-st}\cos(t)\,dt = e^{-st}\left(\sin(t) - s\,\cos(t)\right) - s^2\int_0^\infty e^{-st}\cos(t)\,dt$$

so that

$$(1+s^2)\int_0^\infty e^{-st}\cos(t)\,dt = e^{-st}\left(\sin(t) - s\,\cos(t)\right)$$

and:

$$\int_0^\infty e^{-st}\cos(t)\,dt = \frac{e^{-st}}{s^2+1}\left(\sin(t) - s\,\cos(t)\right)\Big|_0^\infty = \frac{1}{s^2+1}\left(\lim_{T\to\infty}\frac{\sin(T) - s\cos(T)}{e^{sT}} + s\right)$$

To compute the limit, note that:

$$\frac{-1}{e^{sT}} \le \frac{\sin(t)}{e^{sT}} \le \frac{1}{e^{sT}}$$

so that, by the Squeeze Theorem, the overall limit is zero (same if the numerator were $\cos(t)$).

Put everything together now to say that:

$$\int_0^\infty e^{-st}\cos(t)\,dt = \frac{1}{s^2+1}\cdot(0+s) = \frac{s}{s^2+1}$$

5. Show that $y_1(t) = t$, $y_2(t) = t^2$ are linearly independent using the *definition* of linear independence. Compute the Wronskian of $y_1$ and $y_2$: Can they be linearly independent *solutions* to a second order linear differential equation?

$$C_1 t + C_2 t^2 = 0 \text{ for all } t$$

so in particular, if $t = 1$, $C_1 + C_2 = 0$, and if $t = -1$, $-C_1 + C_2 = 0$. Put these two equations together to get that $C_1 = C_2 = 0$.

Therefore, $t$ and $t^2$ are linearly independent. The Wronskian is $t^2$ (Also See Problem 3). They could be independent *solutions* as long as the interval for the solutions does not include zero.

6. Let $\mathbf{x}' = A\mathbf{x}$, where $A$ is given below. Classify the origin (Poincaré Diagram), and give the general analytic solution.

For each matrix, the three numbers that follow are the trace, the determinant and the discriminant (in that order). Remember that the $C_1, C_2$ for $x_1$ are DIFFERENT than the constants for $x_2$. We'll only list $x_1$ below.

(a) $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ 0, 1, -4: CENTER $\quad x_1(t) = C_1\sin(t) + C_2\cos(t)$

(b) $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ 2, 1, 0: DEGENERATE SOURCE $\quad x_1(t) = e^t(C_1 + C_2 t)$

(c) $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ 1, -2, 9: SADDLE $\quad x_1(t) = C_1 e^{2t} + C_2 e^{-t}$

(d) $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$ -2, 5, -16: SPIRAL SINK $\quad x_1(t) = e^{-t}\left(C_1\sin(2t) + C_2\cos(2t)\right)$

(e) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ -2, 0, 4: LINE OF STABLE FIXED POINTS $\quad x_1(t) = C_1 + C_2 e^{-2t}$

7. Let $y''' - y' = te^{-t} + 2\cos(t)$. First, use our ansatz to find the characteristic equation for the third order homogeneous equation. Determine a suitable form for the particular solution, $y_p$ using Undetermined Coefficients. Do not solve for the coeffs.

   The ansatz was $y = e^{rt}$, so that $y' = re^{rt}$ and $y''' = r^3e^{rt}$. Therefore, the homogeneous equation becomes:

$$y''' - y' = 0 \Rightarrow e^{rt}\left(r^3 - r\right) = 0$$

   so that $r(r^2-1) = 0$. Therefore, $r = 0$, $r = \pm 1$. Extrapolating from the second order differential equation, we expect the homogeneous solution to be:

$$y_h = C_1 + C_2e^t + C_3e^{-t}$$

   and the form for the particular solution (break into two pieces):

$$y_{p_1} = (At+B)e^{-t} \Rightarrow y_{p_1} = t(At+B)e^{-t}$$

   and

$$y_{p_2} = A\cos(t) + B\sin(t)$$

8. Write the differential equation associated with *Resonance* and *Beating*. Discuss under what conditions we can expect each type of behavior.

$$y'' + \omega^2 y = F\cos(\omega_0 t)$$

   Beating: If $|\omega - \omega_0|$ is small.

   Resonance: If $\omega = \omega_0$

9. Suppose that we have a mass-spring system modelled by the differential equation

$$x'' + 2x' + x = 0, x(0) = 2, x'(0) = -3$$

   Find the solution, and determine whether the mass ever crosses $x = 0$. If it does, determine the velocity at that instant. See if it crosses if the velocity is cut in half.

   The solution is: $x(t) = e^{-t}(2-t)$, which crosses $x = 0$ when $t = 2$. If we half the initial velocity, the solution is: $x(t) = e^{-t}\left(2 + \frac{1}{2}t\right)$, which does not cross $x = 0$ in positive time.

10. Let $y(x)$ be a power series solution to $(1-x)y'' + y = 0,\ x_0 = 0$. Find the recurrence relation, and write the solution to 6th order.

    At $x_0 = 0$, we have $y^{(k+2)} = ky^{(k+1)} - y^{(k)}$. Using $C_k = k!y^k$, we rewrite this and:

$$(k+2)!C_{k+2} = (k+1)!\,k\,C_{k+1} + k!\,C_k \Rightarrow C_{k+2} = \frac{k}{k+2}C_{k+1} - \frac{1}{(k+1)(k+2)}C_k$$

    Use this to find $C_6, C_5, C_4, C_3, C_2$ in terms of $C_0$ and $C_1$:

$$y(x) = C_0 + C_1x - \frac{1}{2}C_0x^2 - \frac{1}{6}\left(C_0 + C_1\right)x^3 - \frac{1}{24}\left(C_0 + 2C_1\right)x^4 + \left(-\frac{1}{60}C_0 - \frac{1}{24}C_1\right)x^5 + \left(-\frac{7}{720}C_0 - \frac{1}{40}C_1\right)x^6 + \ldots$$

    (Note: On the exam, I would give you initial conditions so you wouldn't have to compute these symbolically in terms of $C_0, C_1$)

11. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write the solution to 6th order.

$$y^{(k+2)} = y^{(k+1)} + (k+1)y^{(k)}$$

    Substitution:

$$(k+2)!\,C_{k+2} = (k+1)!\,C_{k+1} + (k+1)!\,C_k \Rightarrow C_{k+2} = \frac{1}{k+2}\left(C_{k+1} + C_k\right)$$

We get:

$$y(x) = C_0 + C_1(x-1) + \frac{1}{2}(C_0 + C_1)(x-1)^2 + \frac{1}{6}(C_0 + 3C_1)(x-1)^3 + \frac{1}{12}(2C_0 + 3C_1)(x-1)^4 +$$

$$\left(\frac{C_0}{15} + \frac{3C_1}{20}\right)(x-1)^5 + \left(\frac{7C_0}{180} + \frac{C_1}{15}\right)(x-1)^6 + \ldots$$

12. Let $x' = \sin(y)$, $y' = \sin(x)$ Find all equilibria, and classify the stability.

The equilibria satisfy $\sin(y) = 0$ and $\sin(x) = 0$. Therefore, the equilibria are at the points

$$(k_1\pi, k_2\pi), \text{ where } k_1, k_2 \text{ are integers}$$

The Jacobian is:

$$\begin{bmatrix} 0 & \cos(y) \\ \cos(x) & 0 \end{bmatrix}$$

Evaluating at an integer multiple of $\pi$ will yield one of four matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In the middle two cases, we get CENTERS, in either the first or last we get SADDLES. Therefore, the phase plot is a "quilt" of centers and saddles!

13. Analyze how the origin changes classification with respect to $\alpha$ if:

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ -\alpha & -2 \end{pmatrix}\mathbf{x}$$

In this case, the trace is $-1$, the determinant is $\alpha^2 - 2$, and the discriminant is $1 - 4(\alpha^2 - 2) = 9 - 4\alpha^2$.

The sign of the determinant is: $\begin{cases} - & \text{if } -\sqrt{2} < \alpha < \sqrt{2} \\ + & \text{if } \alpha < -\sqrt{2} \text{ or } \alpha > \sqrt{2} \end{cases}$

The sign of the discriminant is: $\begin{cases} + & \text{if } -\frac{3}{2} < \alpha < \frac{3}{2} \\ - & \text{if } \alpha < -\frac{3}{2} \text{ or } \alpha > \frac{3}{2} \end{cases}$

It's easiest to summarize on a number line:

| | | | | | |
|---|---|---|---|---|---|
| det | $+$ | $+$ | $-$ | $+$ | $+$ |
| $\Delta$ | $-$ | $+$ | $+$ | $+$ | $-$ |
| $\alpha$ | $< -3/2$ | $-3/2 < \alpha < \sqrt{2}$ | $-\sqrt{2}\alpha < \sqrt{2}$ | $\sqrt{2} < \alpha < 3/2$ | $> 3/2$ |
| | *SPIRAL SINK* | *SINK* | *SADDLE* | *SINK* | *SPIRAL SINK* |

14. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

$$\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = 3\int_0^2 \mathrm{e}^{-st}\,dt + \int_2^\infty (6-t)\mathrm{e}^{-st}\,dt =$$

$$\frac{3}{s}(1 - \mathrm{e}^{-2s}) + \frac{\mathrm{e}^{-2s}}{s^2}(4s - 1)$$

15. Determine the Laplace transform:

(a) $t^2\mathrm{e}^{-9t} \Rightarrow \dfrac{2}{(s+9)^3}$

(b) $e^{2t} - t^3 - \sin(5t) \Rightarrow \dfrac{1}{s-2} - \dfrac{6}{s^4} - \dfrac{5}{s^2+25}$

(c) $u_5(t)(t-5)^4 \Rightarrow \dfrac{24e^{-5s}}{s^5}$

(d) $e^{3t}\sin(4t) \Rightarrow \dfrac{4}{(s-3)^2+16}$

(e) $e^t\delta(t-3) \Rightarrow e^{-3s+3}$

(f) $t^2u_4(t) \Rightarrow e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$
Note: Let $f(t-4) = t^2$, so that $f(t) = (t+4)^2 = t^2 + 8t + 16$, and $F(s) = \frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}$.

16. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$. Rewrite: $2\cdot\dfrac{s-2}{(s-2)^2+2} + \dfrac{3}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{(s-2)^2+2}$ The inverse is then $e^{2t}\left(2\cos(\sqrt{2}t + \dfrac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right)$

(b) $\dfrac{7}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2e^{-3t}$

(c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$. You might rewrite this as $e^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s+2} + \frac{2}{s-1}$$

Now, $h(t) = 2e^{-2t} + 2e^t$, and the solution is $u_2(t)h(t-2)$.

(d) $\dfrac{3s-2}{(s-4)^2-3}$ We might rewrite this as:

$$3\cdot\frac{s-4}{(s-4)^2-3} + \frac{10}{\sqrt{3}}\cdot\frac{\sqrt{3}}{(s-4)^2-3} = 3F(s-4) + \frac{10}{\sqrt{3}}G(s-4)$$

where $F(s) = \dfrac{s}{s^2-3}, G(s) = \dfrac{\sqrt{3}}{s^2-3}$. The inverse is (Item 14 from the Table):

$$e^{4t}\left(3f(t) + \frac{10}{\sqrt{3}}g(t)\right) = e^{4t}\left(3\cosh(\sqrt{3}t) + \frac{10}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$$

17. Solve the given initial value problems using Laplace transforms:

(a) $y'' + 2y' + 2y = 4t$, $y(0) = 0$, $y'(0) = -1$. The Laplace transform:

$$Y(s) = \frac{4-s^2}{s^2(s^2+2s+2)} = -\frac{2}{s} + \frac{2}{s^2} + \frac{2s+1}{s^2+2s+2} = -\frac{2}{s} + \frac{2}{s^2} + 2\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

so that

$$y(t) = -2 + 2t + e^{-t}\left(2\cos(t) - \sin(t)\right)$$

(b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$. Expanding, we get

$$Y(s) = \frac{10}{(s-2)(s^2+9)} - \frac{s-5}{s^2+9} = \frac{10}{13}\cdot\frac{1}{s-2} - \frac{23}{13}\cdot\frac{s}{s^2+9} + \frac{15}{13}\cdot\frac{3}{s^2+9}$$

so that

$$y(t) = \frac{10}{13}e^{2t} - \frac{23}{13}\cos(3t) + \frac{15}{13}\sin(3t)$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$ Use partial fractions:

$$Y(s) = -\frac{1}{4}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{s-3} + e^{-s}\left(-\frac{1}{3}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s+1} + \frac{1}{12}\cdot\frac{1}{s-3}\right)$$

Think of this second term as $e^{-}s \cdot H(S)$, where

$$h(t) = -\frac{1}{3} + \frac{1}{4}e^{-t} + \frac{1}{12}e^{3t}$$

and the solution is:

$$y(t) = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t} + u_1(t)h(t-1)$$

(d) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$. Recall the hint given in class to use Table Entry 16 and leave your answer in integral form:

$$Y(s) = \frac{2}{(s-1)^3(s-2)^2} = 2 \cdot \frac{2}{(s-1)^3} \cdot \frac{1}{(s-2)^2} = F(s)G(S)$$

Where $f(t) = t^2 e^t$ and $g(t) = te^{2t}$. Therefore,

$$y(t) = \int_0^t (t-\tau)^2 e^{t-\tau} \cdot \tau e^{2\tau}\, d\tau$$

18. Evaluate: $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt = \sin(3\pi/2) = -1$

19. If $y'(t) = \delta(t-c)$, what is $y(t)$? Using Laplace,

$$sY - y(0) = e^{-cs} \Rightarrow Y = e^{-cs} \cdot \frac{1}{s} + y(0) \cdot \frac{1}{s} \Rightarrow y(t) = u_c(t) + y(0) \qquad \text{or simply } y(t) = u_c(t)$$

20. What was the *ansatz* we used to obtain the characteristic equation? $y(t) = e^{rt}$

21. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' - 0.5y = e^{2t}$ $\quad y(0) = 1$. This is a linear (integrating factor) differential equation; the solution will be valid for all time $t$.

$$y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{\frac{1}{2}t}$$

(b) $y'' + 4y' + 5y = 0$, $\quad y(0) = 1, y'(0) = 0$ This is a linear second order with constant coefficients; the solution will be valid for all time t.

$$y(t) = e^{-2t}\left(2\sin(t) + \cos(t)\right)$$

(c) $y' = 1 + y^2$. This is a separable differential equation. with $f(y) = 1 + y^2$. The functions $f$ $f_y$ are continuous for all $t$ and $y$, so existence and uniqueness applies. We have to solve the d.e. to find the interval:

$$\int \frac{1}{1+y^2}\, dy = t + C \Rightarrow \tan^{-1}(y) = t + C \Rightarrow y = \tan(t+C)$$

The solution is only valid for $\frac{-\pi}{2} \le t + C \le \frac{\pi}{2}$.

(d) $y' = \frac{1}{2}y(3-y)$. Similar to the previous problem (this is separable)

$$\int \frac{1}{3} \cdot \frac{1}{y} + \frac{1}{3} \cdot \frac{1}{3-y}\, dy = \frac{1}{2}t + C \Rightarrow \ln|y| - \ln|3-y| = \frac{3}{2}t + C_2$$

Now solve for y:

$$\frac{y}{3-y} = Ae^{3/2t} \Rightarrow y = \frac{3}{1 + Be^{-3/2t}}$$

If the initial condition is positive, this is valid for all time.

(e) $\sin(2x)\mathrm{dx} + \cos(3y)\mathrm{dy} = 0$. You can treat this as a separable differential equation. We get:

$$\sin(3y) = \frac{3}{2}\cos(2x) + C$$

If we were to solve for $y$, we would see that the expression

$$\frac{3}{2}\cos(2x) + C$$

must be between $-1$ and 1 (which is the domain of the inverse sine). That is enough of a description for the review.

(f) $y'' + 2y' + y = 2\mathrm{e}^{-t}, \quad y(0) = 0, y'(0) = 1$

$$y(t) = \mathrm{e}^{-t}\left(t + t^2\right)$$

This solution is valid for all t.

(g) $y' = xy^2$

$$\int y^{-2}\,dy = \frac{1}{2}x^2 + C \Rightarrow -\frac{1}{y} = \frac{x^2 + C_2}{2} \Rightarrow y = \frac{-2}{x^2 + C_2}$$

The interval for which the solution will be valid will depend on if $C_2 > 0$ (the solution will be valid for all $x$), or if $C_2 < 0$ (there will be a vertical asymptote where $x = \pm\sqrt{-C_2}$)

(h) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$ This is an exact equation:

$$\frac{\partial}{\partial y}(2xy^2 + 2y) = 4xy + 2 = \frac{\partial}{\partial x}(2x^2y + 2x)$$

Recall that the solution will be (implicit) $F(x, y) = C$, where

$$F_x = 2xy^2 + 2y \Rightarrow F(x, y) = x^2y^2 + 2xy + h(x)$$

and

$$F_y = 2x^2y + 2x \Rightarrow F(x, y) = x^2y^2 + 2xy + g(y)$$

Comparing, we see $F(x, y) = x^2y^2 + 2xy$, and the implicit solution is:

$$x^2y^2 + 2xy = C$$

Here we will not be able to give an interval on which the solution is valid unless we isolate $y$, although we would have a requirement that $2x^2y + 2x \neq 0$, so that $y'$ would be defined.

(i) $y'' + 4y = t^2 + 3\mathrm{e}^t, y(0) = 0, y'(0) = 1$.

$$y(t) = \frac{1}{5}\sin(2t) - \frac{19}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}\mathrm{e}^t$$

The solution is valid for all $t$.

22. Suppose $y' = -ky(y - 1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of $y$ on the direction field, paying particular attention to where $y$ is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

See the figure online.

23. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of $y$. Hint: Determine the interval for which the solution is valid.

This is separable: $y' = y^2(2 + x) \Rightarrow y^{-2}\,dy = (2 + x)\,dx$, so

$$y(t) = \frac{-2}{t^2 + 4t - 2}$$

This has vertical asymptotes at $t = -2 \pm \sqrt{6}$, so that the solution is valid only when $-2 - \sqrt{6} < t < -2 + \sqrt{6}$, or when $t$ is approximately between $-4.45$ and $0.45$. Between these vertical asymptotes, $y$ has a minimum where its derivative is 0,

$$y' = y^2(2 + x) = 0 \Rightarrow y = 0 \text{ or } x = -2$$

From our solution, we see that $y \neq 0$, so the solution is $x = -2$.

24. If $y(t)$ is a population at time $t$, what is the model for "exponential growth"? What is the model for growth with a "carrying capacity" in the environment? (Recall our Rabbit-Fox model).

Exponential Growth: $y' = ky$

Carrying Capacity (Logistic Equation): $y' = y(k_1 - k_2 y)$

25. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

$$y(t) = \frac{-y_0}{y_0 t^2 - 1}$$

If $y_0 > 0$, then the solution will only be valid between $\pm\frac{1}{\sqrt{y_0}}$. If $y_0 < 0$, the solution will be valid for all $t$.

26. For each nonlinear system, find and classify the equilibria:

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1+2y \\ 1-3x^2 \end{bmatrix}$

$$\left(\frac{1}{\sqrt{3}}, -\frac{1}{2}\right) \Rightarrow \text{ CENTER}$$
$$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{2}\right) \Rightarrow \text{ SADDLE}$$

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1-y \\ x^2-y^2 \end{bmatrix}$

$$(1,1) \Rightarrow \text{ SPIRAL SINK}$$
$$(-1,1) \Rightarrow \text{ SADDLE}$$

(c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+x^2+y^2 \\ y(1-x) \end{bmatrix}$

$$(0,0) \Rightarrow \text{ SOURCE}$$
$$(-1,0) \Rightarrow \text{ SADDLE}$$

(d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-y^2 \\ y-x^2 \end{bmatrix}$

$$(0,0) \Rightarrow \text{ SOURCE}$$
$$(1,1) \Rightarrow \text{ SADDLE}$$

27. Be sure to know the Existence and Uniqueness Theorem for $y' = f(t,y)$ (pg 66) and for linear equations, $y'' + p(t)y' + q(t)y = f(t)$ (pg 138).