# Laplace Transforms Review Solutions

1. Compute transforms from the definition:

   (a) $\displaystyle\int_0^2 3e^{-st}\,dt + \int_2^\infty (6-t)e^{-st}\,dt = \frac{3}{s} - \frac{3}{s}e^{-2s} + e^{-2s}\left(\frac{4}{s} - \frac{1}{s^2}\right)$

   (b) $\displaystyle\int_0^5 e^{-t}e^{-st}\,dt - \int_5^\infty e^{-st}\,dt = \frac{1-e^{-5(s+1)}}{s+1} - \frac{e^{-5s}}{s}$

2. (a) $f(t) = 3(1-u(t-2)) + (6-t)u(t-2)$ or $3(u(t)-u(t-2)) + (6-t)u(t-2)$
   For the Laplace Transform, use $f(t-a)u(t-a)$. If $f(t-2)=3$, then $f(t)=3$. If $f(t)=6-t$, then $f(t)=6-(t+2)=-t+4$.

   (b) $f(t) = e^{-t}(1-u(t-5)) - u(t-5)$. For the Laplace transform, if $f(t-5)=e^{-t}$, then $f(t) = e^{-(t+5)} = e^{-5}e^{-t}$

3. Compute transforms (using the table)

   (a) $\frac{2}{(s+9)^3}$

   (b) $\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$

   (c) $e^{-5s}\frac{4!}{s^5}$

   (d) $\frac{4}{(s-3)^2+16}$

   (e) Use $e^{at}f(t) \to F(s-a)$, with $f(t)=\delta(t-3)$, so $e^{-3(s-1)}$.

   (f) Let $f(t-4)=t^2$, so $f(t)=(t+4)^2$: $e^{-4s}\left(\frac{2}{s^3} + \frac{8}{2^2} + \frac{16}{s}\right)$

4. Invert the transforms:

   (a) First rewrite as $\frac{2s-1}{(s-2)^2+2}$, so $2e^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}e^{2t}\sin(\sqrt{2}t)$

   (b) Via partial fractions: $e^{-t} - 3e^{-3t} + 3e^{2t}$

   (c) $\frac{7}{2}e^{-3t}t^2$

   (d) $u(t-2)[2e^{-2(t-2)} + 2e^{t-2}]$

   (e) Rewrite to get: $\frac{3}{2}\left(\frac{(s-4)}{(s-4)^2-11} + \frac{10}{3\sqrt{11}}\frac{\sqrt{11}}{(s-4)^2-11}\right)$, so we get: $\frac{3}{2}e^{4t}\cosh(\sqrt{11}t) + \frac{5}{\sqrt{11}}e^{4t}\sinh(\sqrt{11}t)$

   (f) Write as $e^{-2s}H(s)$.

   $$H(s) = \frac{1}{s^2+2s-2} = \frac{1}{(s+1)^2-3} = F(s+1)$$

   where $F(s) = 1/(s^2-3)$. This gives $f(t) = 1/\sqrt{3}\ \sinh(\sqrt{3}t)$. Final answer:

   $$u(t-2)\cdot\frac{1}{\sqrt{3}}e^{-(t-2)}\sinh(\sqrt{3}(t-2))$$

5. Solve the diff. eqn:

   (a) $e^{2t} - e^{5t}$

   (b) $-3e^{-3t} + te^{-3t}$

(c) You might first write:

$$Y = \frac{s^3+4s+2}{s^3(s^2+2s+2)} = -\frac{3}{2}\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{3}{2}\frac{s+2}{s^2+2s+2}$$

$-\frac{3}{2} + t + \frac{1}{2}t^2 - \frac{1}{2}\mathrm{e}^{-t}\sin(t) + \frac{3}{2}\,\mathrm{e}^{-t}\cos(t)$

(d) $\frac{10}{13}\mathrm{e}^{2t} - \frac{23}{13}\cos(3t) + \frac{15}{13}\sin(3t)$

(e) Let $h(t) = \frac{-1}{3} + \frac{1}{12}\mathrm{e}^{3t} + \frac{1}{4}\mathrm{e}^{-t}$ Then the solution is: $y(t) = u(t-1)h(t-1) - \frac{1}{4}\mathrm{e}^{3t} + \frac{1}{4}\mathrm{e}^{-t}$.

(f) We need to invert $\frac{2}{(s-1)^3(s-2)^2} = \frac{6}{s-1} + \frac{4}{(s-1)^2} + \frac{2}{(s-1)^3} - \frac{6}{(s-2)} + \frac{2}{(s-2)^2}$, which is $\mathrm{e}^{2t}(2t-6) + \mathrm{e}^t(t^2+4t+6)$

(g) $\frac{1}{2}\sin(2t) + \frac{1}{2}u(t-\pi/2)\sin(2(t-\frac{\pi}{2}))$

(h) $y(t) = \sum_{k=1}^{\infty} u_{2\pi k}(t)\sin(t)$ Note that this is:

$$y(t) = \begin{cases} \sin(t), & 0 \le t < 2\pi \\ 2\sin(t), & 2\pi \le t < 4\pi \\ 3\sin(t), & 4\pi \le t < 6\pi \\ \vdots & \vdots \end{cases}$$

6. Evaluate: $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\,dt$

We know that $\int_{-\infty}^{\infty} \delta(t-c)f(t)\,dt = f(c)$. Therefore, for $c > 0$,

$$\int_0^\infty \sin(3t)\delta(t-\pi/2)\,dt = \sin\left(\frac{3\pi}{2}\right) = -1$$

7. $\mathcal{L}(t * \sin(t)) = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \frac{1}{s^2} - \frac{1}{s^2-1}$ and the inverse laplace of that is $t - \sin(t)$.

8. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to solve:

$$y'' + 3ty' - 6y = 1, \quad y(0) = 0, \quad y'(0) = 0$$

From the table, $\mathcal{L}(ty') = -Y(s) - sY'(s)$. Take the Laplace transform, and:

$$s^2Y(S) + 3(-Y - sY') - 6Y = \frac{1}{s}$$

$$\Rightarrow Y' + \left(\frac{s^2-9}{-3s}\right)Y = -\frac{1}{3s}$$

Use the method of the integrating factor:

$$\int \frac{s^2-9}{-3s}\,ds = \int -\frac{1}{3}s + \frac{3}{s}\,ds = -\frac{1}{6}s^2 + 3\ln(s)$$

Now the integrating factor is $\mathrm{e}^{(-1/6)s^2 + \ln(s^3)} = s^3\mathrm{e}^{(-1/6)s^2}$:

$$(s^3\mathrm{e}^{(-1/6)s^2}Y)' = -\frac{1}{3s}s^3\mathrm{e}^{(-1/6)s^2} = -\frac{1}{3}s\mathrm{e}^{(-1/6)s^2}$$

so that

$$s^3\mathrm{e}^{(-1/6)s^2}Y = \mathrm{e}^{(-1/6)s^2} \Rightarrow Y(s) = \frac{1}{s^3}$$

Therefore, $y(t) = \frac{1}{2}t^2$.

9. Characterize all solutions: We solve by our old method of getting the homogeneous and particular solutions.

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t), & 0 \le t < 1 \\ c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}, t \ge 1 \end{cases}$$

10. Define the delta function:

$$\delta(t-c) = \lim_{h\to 0} d_h(t-c)$$

where

$$d_h(t-c) = \begin{cases} \frac{1}{2h}, & c-h < t < c+h \\ 0, & \text{otherwise} \end{cases}$$

11. If $y'(t) = \delta(t-c)$, what is $y(t)$?

$$sY = \mathrm{e}^{-cs} \to Y = \frac{\mathrm{e}^{-cs}}{s} \to y(t) = u(t-c)$$

12. **Tank Mixing Problems:** Page 117-118, Exercises 5-8. Page 122, Exercise 16.

13. The following system of D.E.s describes the interaction of a population of predators with a population of prey. (a) Which is the predator, and which is the prey (and why)? (b) Find all the equilibrium solutions.

$$\begin{aligned} \dot{x} &= x(-1+3y) \\ \dot{y} &= y(0.4-2x) \end{aligned}$$

The prey is $y$, predators are $x$ (Note that in the absence of the other, $x$ decreases but $y$ increases). The equilibria are where the two derivatives are equal to zero (at the same time). This is where $x=0, y=0$ or $x=1/5, y=1/3$.

For the spring/mass equations below, the force of gravity is 32ft/sec$^2$.

14. A 4-foot spring measures 8 feet long after a mass weighing 8 lbs. is attached to it. The medium through which the mass moves offers a damping force equivalent to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5ft/sec.

For $mx'' + \gamma x' + kx = 0$, we have that $m = \frac{8}{32}$, $k = 2$ (from $mg = kL \Rightarrow 8 = k \cdot 4$) and $\gamma = \sqrt{2}$. The D.E. is then:

$$\frac{1}{4}x'' + \sqrt{2}x' + 2x = 0, x(0) = 0, x'(0) = 5$$

and the solution is $x(t) = 5t\mathrm{e}^{-2\sqrt{2}t}$.

15. Let $mx'' + \gamma x' + kx = F_0\cos(\omega t)$ be our mass/spring model. (i) What are the conditions on $m, \gamma, k$ and $\omega$ so that the system is *resonant*? (ii) True or False, and give explicit (mathematical) reasons why: If there is resistance, then the homogeneous part of the solution will tend to zero. (This was also called the transient part of the solution)

(i) For resonance, $\gamma = 0$ and $\frac{k}{m} = \sqrt{\omega}$. (ii) If $\gamma > 0$, we considered $x'' + 2\lambda x' + \omega^2 x$ (different $\omega$), and looked at the homogeneous solutions. The solutions to the characteristic equation were $-\lambda \pm \sqrt{\lambda^2 - \omega^2}$. This gives solutions in which $e^{-\lambda t}$ can be factored out front, which will cause the solutions to die off as $t \to \infty$.

16. Assume no damping, and that a mass weighing two pounds stretches a spring 6 inches. The mass is released from equilibrium with an upward velocity of $1 ft/sec$. (a) Write the equation of motion, and solve. (b) Suppose we hit the mass with a hammer at time $a > 0$ (use $\delta(t-a)$). Model this and re-solve using Laplace transforms. (Optional: Find a time $a$ so that when we hit the mass with the hammer, it stops all motion).

    For part (a) and the spring constant, convert 6 inches to 0.5 feet to get: $x'' + 64x = 0$, $x(0) = 0, x'(0) = -1$. From this, $x(t) = -\frac{1}{8}\sin(8t)$.

    If we hit the spring with a hammer,

$$x'' + 64x = \delta(t-a) \Rightarrow x(t) = -\frac{1}{8}\sin(8t) + \frac{1}{8}U(t-a)\sin(8(t-a))$$

    For what time(s) $a$ does $\sin(8(t-a)) = \sin(8t)$? The period of $\sin(8t)$ is $\pi/4$, so if we shift by any multiple of that, it would work. So, $a = n \cdot \frac{\pi}{4}$.

17. Let $x, y, z$ be three populations of animals with the following properties:

    (a) In the absence of $y, z$, the population of $x$ grows exponentially.
    (b) In the absence of $x, z$, the population of $y$ follows logistic growth.
    (c) In the absence of $x, y$, the population $z$ declines exponentially.
    (d) Populations $x, y$ compete for the same resources, so each of their populations will decrease in proportion to the number of interactions between them.
    (e) Populations $x, y$ are food for the predator $z$, so each of their populations will decrease in proportion to the number of interactions between them (assume only $x - z$ and $x - y$ interactions). Also, the population of $z$ will increase in proportion to the number of the interactions (again, consider only $x - z$ and $x - y$).

    Build a system of differential equations that will model how the populations $x, y, z$ will change over time. (Do not solve the system)

$$\begin{aligned} \dot{x} &= k_1 x - k_5 xy - k_7 xz \\ \dot{y} &= y(k_2 + k_3 y) - k_6 xy - k_8 yz \\ \dot{z} &= -k_4 z + k_9 xz + k_{10} yz \end{aligned}$$

18. Epidemic Models of the Onset of Social Activities (EMOSA). Epidemic models have been used to model different sorts of social activity. In this question, we develop a model used for the transition from "virgin" to "nonvirgin"[1]. Let $P_m$, $P_f$ be the proportion of "nonvirgins" males, females (respectively) in a given heterosexual population (so that $1 - P_m, 1 - P_f$ are the proportions of "virgins"). The model assumes the following:

[1]Modified slightly from: "Social contagion, adolescent sexual behavior, and pregnancy: A nonlinear dynamic EMOSA model, Rodgers, J., Rowe, D., Buster, M. *Developmental Psychology*, 1998 34(5), 1096-1113

(a) The proportion of nonvirgin males changes at a rate proportional to the number of virgin-virgin interactions, and proportional to the number of virgin male- nonvirgin female interactions.

(b) Similarly, the proportion of nonvirgin females changes at a rate proportional to the number of virgin-virgin interactions and proportional to the number of virgin female - nonvirgin male interactions.

(i) Build the model that follows from these assumptions. (ii) Determine the equilibrium solutions.

(Side remark: Its interesting to consider how models of disease might be modified to explain social behaviors such as the ones considered here. These questions are very current in sociology and psychology.)

$$
\begin{aligned}
P_m' &= k_1(1-P_m)(1-P_f) + k_2(1-P_m)P_f = (1-P_m)[k_1(1-P_f) + k_2P_f] \\
P_f' &= k_3(1-P_m)(1-P_f) + k_4(1-P_f)P_m = (1-P_f)[k_3(1-P_m) + k_4P_m]
\end{aligned}
$$

The equilibria are at $(1, 1)$ and

$$
\left( \frac{k_1}{k_1 - k_2}, \frac{k_3}{k_3 - k_4} \right)
$$