Solutions: Section 2.4

1. In Problems 1, 3 and 5 we want to find a single connected interval in which both $p(t)$ and $g(t)$ are continuous.

   (a) Problem 1: In standard form,

   $$y' + \frac{\ln(t)}{t-3}y = \frac{2t}{t-3}$$

   The restriction on the log makes $t > 0$, and the $t - 3$ makes $t \neq 3$. Together, this breaks the number line into two possibilities: Either $0 < t < 3$ or $t > 3$. The initial time is $t_0 = 1$, which makes us choose the first interval, $0 < t < 3$.

   (b) Problem 3: Similarly,

   $$y' = \tan(t)y = \sin(t)$$

   The tangent function has vertical asymptotes, which separates the possible intervals that could be used. Some possibilities:

   $$-\frac{\pi}{2} < t < \frac{\pi}{2}, \text{ or } \frac{\pi}{2} < t < \frac{3\pi}{2}, \text{ or } \frac{3\pi}{2} < t < \frac{5\pi}{2}, \text{ etc.}$$

   Looking at the initial condition, $t_0 = \pi$, so choose the interval with this value: $\pi/2 < t < 3\pi/2$.

   (c) Problem 5:

   $$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$$

   The only restriction here is that $t \neq \pm 2$. This splits the possible intervals to: $t < -2, -2 < t < 2$, or $t > 2$. Since the initial time is $t_0 = 1$, we choose the middle interval, $-2 < t < 2$.

2. Problem 7: In this case,

   $$f(t, y) = \frac{t - y}{2t + 5y}$$

   We see that f is continuous away from the line $2t + 5y = 0$, which splits the plane into two pieces (above/below the line).

   Check the partial derivative with respect to y:

   $$\frac{\partial f}{\partial y} = \frac{-7t}{(2t + 5y)^2}$$

   so we still have the same restriction.

3. Problem 9:

$$f(t,y) = \frac{\ln|ty|}{1-t^2+y^2}$$

This is continuous away from $ty = 0$ (away from the $t-$ and $y-$axis, and where $1-t^2+y^2 \neq 0$. The points where this is zero forms a hyperbola in the plane. Compute the partial derivative:

$$\frac{\partial f}{\partial y} = \frac{\frac{t}{ty}(1-t^2+y^2) - \ln|ty|(2y)}{(1-t^2+y^2)^2}$$

This does not add any new restrictions.

4. Problem 13: Solve $y' = -4t/y$, $y(0) = y_0$.

This is separable, so

$$\int y\,dy = -4\int t\,dt \Rightarrow \frac{1}{2}y^2 = -2t^2 + C$$

With the initial condition,

$$\frac{1}{2}y_0^2 = 0 + C \Rightarrow \frac{1}{2}y^2 = -2t^2 + \frac{1}{2}y_0^2$$

We can simplify a bit:

$$y^2 = -4t^2 + y_0^2 \quad \Rightarrow \quad y(t) = \pm\sqrt{y_0^2 - 4t^2}$$

We would choose the positive root if $y_0 > 0$, or the negative root if $y_0 < 0$. Furthermore, for the square root to be defined,

$$y_0^2 - 4t^2 > 0 \quad \Rightarrow \quad (y_0 - 2t)(y_0 + 2t) > 0$$

This separates the plane into three regions. We choose the middle since $t_0 = 0$. Assuming $y_0 > 0$, we have:

$$-\frac{y_0}{2} \leq t \leq \frac{y_0}{2}$$

(If $y_0 < 0$, flip the inequalities).

5. Problem 14: $y' = 2ty^2$, $y(0) = y_0$. Separable:

$$\int y^{-2}\,dy = \int 2t\,dt \quad \Rightarrow \quad -\frac{1}{y} = t^2 + C$$

With the initial condition, $C = -1/y_0$, so:

$$y(t) = \frac{1}{(1/y_0) - t^2} = \frac{y_0}{1 - y_0 t^2}$$

This is valid as long as $y_0 \neq 0$. What if it is? Then we see that $y(t) = 0$ is the unique solution.

If $y_0 \neq 0$, then we continue by looking at where

$$1 - y_0 t^2 = 0 \qquad \Rightarrow \qquad t = \pm \frac{1}{\sqrt{y_0}}$$

This is valid only if $y_0 > 0$. If $y_0 < 0$, then the denominator, $1 - y_0 t^2$ is never zero (for any t). Thus, if $y_0 < 0$, the solution that we previously obtained is valid for all t.

The last case is where $y_0 > 0$. Since the initial time is $t_0 = 0$, then the solution $y(t)$ is only valid for:

$$-\frac{1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$$

Summary: In this homework problem, we saw that the time interval on which the solution is valid depended greatly on the initial value of y,

- If $y_0 < 0$, $y(t)$ is valid for all time.
- If $y_0 = 0$, $y(t) = 0$ is the solution, valid for all time.
- If $y_0 > 0$, $y(t)$ is valid for a short segment of time, between $\pm 1/\sqrt{y_0}$.

6. Problem 15: $y' + y^3 = 0$, $y(0) = y_0$. This is separable:

$$y' = -y^3 \qquad -y^{-3}\,dy = dt \qquad \frac{1}{2}y^{-2} = t + C$$

This technique is only valid if $y \neq 0$. From the original differential equation, we see that $y(t) = 0$ would be the (unique) solution if $y_0 = 0$. Otherwise, we can solve for C:

$$\frac{1}{2y_0^2} = 0 + C$$

We now want to solve for y explicitly:

$$y^{-2} = 2t + \frac{1}{y_0^2} \qquad y^2 = \frac{1}{2t + \frac{1}{y_0^2}}$$

The main issue here: $2t + \frac{1}{y_0^2} > 0$ in order for us to solve for y. In that case,

$$t > -\frac{1}{2y_0^2}$$

Also, we need to decide on whether to take the positive or negative square root when solving for $y(t)$: This depends on the sign of y_0; if $y_0 > 0$, then take the positive root, otherwise take the negative root.

In any event, we still have the restriction on time.

7. Problem 22: Parts (a) and (b) were done in class.

   The point of part (c) is to show that the second solution, $y_2(t)$ cannot be found as a constant multiple of the first (why? This will become clearer in Chapter 3).

   Clearly, no value of a constant in $ct + c^2$ can give you the function $-\frac{1}{4}t^2$.

8. Problem 23:

   (a) Show that e^{2t} and ce^{2t} (c any constant) are both solutions to the ODE: $y' - 2y = 0$. You can show this directly (by substitution), or by actually solving the DE. You should see that the general solution is $y(t) = Ae^{2t}$

   (b) Show that $\frac{1}{t}$ is a solution to $y' + y^2 = 0$, but $\frac{C}{t}$ is not. You can again show this directly (by substitution), or by actually solving the DE. If you solve it, you should get:

   $$y(t) = \frac{1}{t - C}$$

   (or $y(t) = 0$).

9. Problem 24: To show this, first note that, if $y(t) = \phi(t)$ is a solution to $y' + p(t)y = 0$, then:

   $$\phi' + p(t)\phi = 0$$

   Now substitute $y(t) = c\phi$: $y' = c\phi'$, and:

   $$c\phi' + p(t)c\phi = c\left(\phi' + p(t)\phi\right) = c \cdot 0 = 0$$

10. Problem 25: Same idea as 24. Substitute the expression in to see what you get.

    Assume that y_1 solves $y' + p(t)y = 0$. This means that $y_1' + p(t)y_1' = 0$.

    Assume that y_2 solves $y' + p(t)y = g(t)$. This means that $y_2' + p(t)y_2 = g(t)$.

    Now, substitute $y = y_1 + y_2$, $y' = y_1' + y_2'$ into the DE:

    $$(y_1' + y_2') + p(t)(y_1 + y_2) = (y_1' + p(t)y_1) + (y_2' + p(t)y_2) = 0 + g(t) = g(t)$$