Selected Problems: 3.5

1. Problem 16: Solve the IVP:

   $$y'' - y' + \frac{1}{4}y = 0 \qquad y(0) = 2, \quad y'(0) = b$$

   From the characteristic equation, $r = 1/2$ is a repeated root. Therefore, the general solution is:

   $$y = \mathrm{e}^{t/2}(C_1 + C_2 t)$$

   Solving for the coefficients,

   $$y = \mathrm{e}^{t/2}(2 + (b-1)t)$$

   For large t, the term $(b-1)t\mathrm{e}^{t/2}$ will dominate. Therefore, as $t \to \infty$, if $b > 1$, this term goes to $+\infty$, and if $b < 1$, then this term goes to $-\infty$.

2. Problem 18: Solving the differential equation with the given initial values, we get:

   $$y = \mathrm{e}^{-\frac{2}{3}t}\left(\alpha + \frac{2\alpha - 3}{3}t\right)$$

   By an argument similar to the previous problem, the critical value of α is the one that makes $2\alpha - 3 = 0$. Therefore, $\alpha = \frac{3}{2}$.

3. Problem 20: Done in class (in the more general form of $ay'' + by' + cy = 0$), but let's look at this special case.

   (a) Char Eqn: $r^2 + 2ar + a^2 = 0 \Rightarrow (r+a)^2 = 0$, so $r = -a$ is a double root.

   (b) The Wronskian:

   $$W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt} = C\mathrm{e}^{-2at}$$

   (c) Let $y_1 = \mathrm{e}^{-at}$. Find y_2 using the Wronskian:

   $$\begin{vmatrix} \mathrm{e}^{-at} & y_2 \\ -a\mathrm{e}^{-at} & y_2' \end{vmatrix} = \mathrm{e}^{-at}\left(y_2' + ay_2\right)$$

   Setting the two expressions for the Wronskian equal to each other enables us to find y_2:

   $$\mathrm{e}^{-at}\left(y_2' + ay_2\right) = C\mathrm{e}^{-2at} \quad \Rightarrow \quad y_2' + ay_2 = C\mathrm{e}^{-at}$$

   The integrating factor is e^{at}. Multiply both sides by it, and go through our usual method for solving:

   $$\left(y_2\mathrm{e}^{at}\right)' = C \quad \Rightarrow \quad y_2\mathrm{e}^{at} = Ct + C_2 \quad \Rightarrow \quad y_2 = Ct\mathrm{e}^{-at} + C_2\mathrm{e}^{-at}$$

   The last term is already present as y_1, so we take y_2 to be $t\mathrm{e}^{-at}$.

4. Problem 21: We are told that $\mathrm{e}^{r_1 t}$ and $\mathrm{e}^{r_2 t}$ are solutions to the DE:

   $$ay'' + by' + cy = 0$$

   where $r_1 \neq r_2$.

   Is the following function also a solution?

   $$\phi = \frac{\mathrm{e}^{r_2 t} - \mathrm{e}^{r_1 t}}{r_2 - r_1} = \frac{1}{r_2 - r_1}\mathrm{e}^{r_2 t} - \frac{1}{r_2 - r_1}\mathrm{e}^{r_1 t}$$

   Yes, since it is of the form $C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$.

   Now use L'Hospital's rule to evaluate this solution as $r_2 \to r_1$. Note that we'll be differentiating with respect to the variable that is changing, r_2. Therefore, treat r_1 and t as constants:

   $$\lim_{r_2 \to r_1} \frac{\mathrm{e}^{r_2 t} - \mathrm{e}^{r_1 t}}{r_2 - r_1} = \lim_{r_2 \to r_1} \frac{t\mathrm{e}^{r_2 t} - 0}{1 - 0} = t\mathrm{e}^{r_1 t}$$

5. Problem 38: If a, b, c are positive constants, then all solutions to $ay'' + by' + cy = 0$ tend to zero as $t \to \infty$.

   We need to consider our three cases:

   - If $b^2 - 4ac > 0$ then we need to consider the sign of $-b \pm \sqrt{b^2 - 4ac}$. If a, b, c are positive then

     $$b > \sqrt{b^2 - 4ac}$$

     since we are subtracting from $\sqrt{b^2}$. In that case, r_1 and r_2 are both negative, and

     $$C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t} \to 0 \qquad \text{as } t \to \infty$$

   - If $b^2 - 4ac < 0$, then:

     $$r = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

     And the general solution is:

     $$y = \mathrm{e}^{-(b/2a)t}\left(C_1 \cos\left(\frac{\sqrt{4ac - b^2}}{2a}\right) + C_2 \sin\left(\frac{\sqrt{4ac - b^2}}{2a}\right)\right)$$

     The sum of a sine and cosine are bounded, so the exponential term in front drives the solution to zero as $t \to \infty$.

   - If $b^2 - 4ac = 0$, then $r = -b/2a$ and the general solution is:

     $$y(t) = \mathrm{e}^{-(b/2a)t}(C_1 + C_2 t)$$

     From L'Hospital's rule,

     $$\lim_{t \to \infty} \frac{C_1 + C_2 t}{\mathrm{e}^{(b/2a)t}} = \lim_{t \to \infty} \frac{C_2}{(b/2a)\mathrm{e}^{(b/2a)t}} = 0$$

6. Problem 39 is a special case of Problem 38. Note that if $b = 0$, and a, c are positive, we will always have a complex solution to the characteristic equation,

   $$r = \pm i\,\frac{\sqrt{ac}}{a} \doteq \pm\mu i$$

   So that the general solution is:

   $$C_1\cos(\mu t) + C_2\sin(\mu t)$$

   which is always bounded.

   The second part, with $a > 0, b > 0$ but $c = 0$ means that the solution to the homogeneous equation is $r = 0, -b$. In that case,

   $$y = C_1 + C_2 \mathrm{e}^{-bt}$$

   and the solution goes to C_1 as $t \to \infty$.

   Solve for C_1 in terms of y_0 and y_0' (see the text solution).

7. Problem 40 looks at the case where we have functions as coefficients instead of a, b, c. We are asked to verify that $y = \sin(t)$ is a solution to the DE:

   $$y'' + (k\sin^2(t))y' + (1 - k\cos(t)\sin(t))y = 0$$

   This should be straightforward- Simply substitute and verify.

   In the next part, we are asked to look at the sign of the coefficients to see if the analysis of Problems 38 and 39 still applies.

   The coefficient in front of y'' is 1, which is always positive.

   The coefficient in front of y' is $k\sin^2(t)$, which is non-negative as long as $k > 0$.

   The coefficient in front of y is $1 - k\cos(t)\sin(t)$. This is positive as long as

   $$k\cos(t)\sin(t) < 1$$

   We can see where this is true if we knew the maximum of $\cos(t)\sin(t)$. To get this maximum, use Calc I stuff:

   $$f(t) = \cos(t)\sin(t) \quad \Rightarrow \quad f'(t) = -\sin^2(t) + \cos^2(t) = 0$$

   The maximum (and minimum) occur where

   $$\sin(t) = \pm\cos(t)$$

   On the unit circle, these are $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ (and we could add multiples of 2π). The max occurs where the sine and cosine are both positive, $\sin(\pi/4)\cos(\pi/4) = \frac{1}{2}$.

Going back, we know that:

$$k\cos(t)\sin(t) \leq \frac{k}{2}$$

so if $0 < k < 2$, then $k\cos(t)\sin(t) < 1$, and the coefficient in front of y is positive.

Note: The point here is that the analysis from Problem 38 only applies to DEs with constant coefficients. In this problem, our solution does NOT tend to zero, even thought the coefficients are all positive.