

Exercise Set 2 (replaces Ch 7)

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

Below are two methods by which we can check our answer. Use the one you're most comfortable with.

Method 1: Rewrite the solution in terms of x_1 and x_2 :

$$x_1 = C_1 e^{-t} + 2C_2 e^{2t} \quad x_2 = 2C_1 e^{-t} + C_2 e^{2t}$$

Differentiate:

$$x_1' = -C_1 e^{-t} + 4C_2 e^{2t} \quad x_2' = -2C_1 e^{-t} + 2C_2 e^{2t}$$

Now substitute the expressions for x_1 and x_2 to show that $x_1' = 3x_1 - 2x_2$ and $x_2' = 2x_1 - 2x_2$.

Method 2 (Use matrix algebra):

$$\mathbf{x}'(t) = -C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

and now compute $A\mathbf{x}$

$$\begin{aligned} & \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \left(C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \\ & C_1 e^{-t} \left(\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + C_2 e^{2t} \left(\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \\ & C_1 e^{-t} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \end{aligned}$$

2. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3:

$$(a) \quad A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= -5x_1 - x_2 \end{aligned}$$

We use the substitution from the first equation, $x_2 = \frac{1}{2}(x_1' - x_1)$ into the second equation:

$$\frac{1}{2}(x_1'' - x_1') = -5x_1 - \frac{1}{2}(x_1' - x_1) \quad \Rightarrow \quad x_1'' + 9x_1 = 0$$

We have two complex roots to the characteristic equation, $\lambda = \pm 3i$, so

$$x_1(t) = C_1 \cos(3t) + C_2 \sin(3t) \quad \Rightarrow \quad x_2(t) = \frac{1}{2}(x_1' - x_1)$$

which simplifies to

$$x_2(t) = C_1 \left(-\frac{1}{2} \cos(3t) - \frac{3}{2} \sin(3t) \right) + C_2 \left(\frac{3}{2} \cos(3t) - \frac{1}{2} \sin(3t) \right)$$

For extra practice: The eigenvalues are the same as the roots to our old roots to the characteristic equation, $\lambda = 3i$. Substitute to get the eigenvector:

$$\begin{aligned} v_1 + 2v_2 &= 3iv_1 \\ -5v_1 - v_2 &= 3iv_2 \end{aligned} \quad v_2 = \left(-\frac{1}{2} + \frac{3}{2}i\right)v_1 \quad \mathbf{v} = \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{3}{2}i \end{bmatrix}$$

Now we need $e^{\lambda t}\mathbf{v}$

$$\begin{aligned} &(\cos(3t) + i\sin(3t)) \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{3}{2}i \end{bmatrix} = \\ &\begin{bmatrix} \cos(3t) + i\sin(3t) \\ -\frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t) + i\left(\frac{3}{2}\cos(3t) - \frac{1}{2}\sin(3t)\right) \end{bmatrix} \end{aligned}$$

We recall that

$$\mathbf{x}(t) = C_1 \text{Real}(e^{\lambda t}\mathbf{v}) + C_2 \text{Imag}(e^{\lambda t}\mathbf{v})$$

which in this case is:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} \cos(3t) \\ -\frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(3t) \\ \frac{3}{2}\cos(3t) - \frac{1}{2}\sin(3t) \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ Using the first equation to solve for $x_2 = x'_1 - x_1$, substitute into the second to get:

$$x''_1 - x'_1 = 4x_1 + x'_1 - x_1 \Rightarrow x''_1 - 2x'_1 - 3x_1 = 0$$

From solving the characteristic equation, $r = -1, 3$ so that

$$x_1 = C_1 e^{3t} + C_2 e^{-t}$$

Use the substitution to find $x_2 = x'_1 - x_1$:

$$x_2 = 2C_1 e^{3t} - 2C_2 e^{-t}$$

For extra practice, solving this as a system using eigenvalues and eigenvectors, the eigenvalues are $\lambda = -1, 3$ (same as the r from Chapter 3), and solving for the eigenvectors: $\lambda = -1$, we get:

$$\begin{aligned} 2v_1 + v_2 &= 0 \\ 4v_1 + 2v_2 &= 0 \end{aligned} \Rightarrow v_2 = -2v_1 \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For $\lambda = 3$, we get:

$$\begin{aligned} -2v_1 + v_2 &= 0 \\ 4v_1 - 2v_2 &= 0 \end{aligned} \Rightarrow v_2 = 2v_1 \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Putting these together for the solution (reversing them so they look like the solution we got earlier):

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 = C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

In this case, the second equation is easier to use for the substitution: $x_1 = x_2' + x_2$, so that the first equation becomes:

$$x_1'' + x_2' = 3x_2' + 3x_2 - 4x_2 \Rightarrow x_2'' - 2x_2' + x_2 = 0$$

We have a double root: $r = 1, 1$. Thus,

$$x_2 = e^t (C_1 + C_2 t)$$

Substitute this into the equation for x_1 :

$$x_1 = x_2' + x_2 = e^t (2C_1 + C_2(2t + 1))$$

For extra practice, solving this as a system using eigenvalues and eigenvectors, the eigenvalues are $\lambda = 1, 1$ solving for the eigenvector:

$$\begin{aligned} 2v_1 - 4v_2 &= 0 \\ v_1 - 2v_2 &= 0 \end{aligned} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For the generalized eigenvector \mathbf{w} , we get any solution to the following:

$$\begin{aligned} 2w_1 - 4w_2 &= 2 \\ w_1 - 2w_2 &= 1 \end{aligned} \Rightarrow \begin{aligned} w_1 &= 1 \\ w_2 &= 0 \end{aligned} \Rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The solution is:

$$\mathbf{x} = e^t (C_1 \mathbf{v} + C_2 (t\mathbf{v} + \mathbf{w})) = e^t \left[C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left[t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] \right]$$

3. Suppose we have a system of two differential equations, and the system gave us the eigenvalues and eigenvectors listed. For each, write the general solution to the differential equation:

$$(a) \lambda_1 = -2 \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \lambda_2 = -1 \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

SOLUTION:

$$C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$(b) \lambda = -1 + i \quad \mathbf{v} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix} \text{ We need to compute } e^{\lambda t \mathbf{v}}$$

SOLUTION:

$$e^{\lambda t \mathbf{v}} = e^{-t} e^{it} \begin{bmatrix} 1 + i \\ 2 \end{bmatrix} = e^{-t} (\cos(t) + i \sin(t)) \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

Factor out e^{-t} for a moment, and multiply the rest out:

$$(\cos(t) + i \sin(t)) \begin{bmatrix} 1 + i \\ 2 \end{bmatrix} = \begin{bmatrix} \cos(t) - \sin(t) + i(\cos(t) + \sin(t)) \\ 2 \cos(t) + i2 \sin(t) \end{bmatrix}$$

The solution to the differential equation uses the real and imaginary parts of this (put back in the exponential, too):

$$\mathbf{x} = C_1 e^{-t} \begin{bmatrix} \cos(t) - \sin(t) \\ 2 \cos(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \cos(t) + \sin(t) \\ 2 \sin(t) \end{bmatrix}$$

- (c) $\lambda = 1, 1$ $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where \mathbf{w} is the generalized eigenvector.

$$\mathbf{x} = e^t \left[C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 \left[t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right] \right]$$

4. For each matrix, find the eigenvalues and eigenvectors. If there is only one eigenvector, find the associated generalized eigenvector.

The solution will be to find the characteristic equation, solve for the eigenvalues, then find the eigenvectors. You may use the shortcut using the trace, determinant and discriminant.

(a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \Rightarrow \lambda^2 - 6\lambda + 8 = 0 \quad \lambda = 2, 4$

Use $\lambda = 2$ to find the first eigenvector:

$$\begin{aligned} (5-2)v_1 - v_2 &= 0 \\ 3v_1 + (1-2)v_2 &= 0 \end{aligned} \Rightarrow v_2 = 3v_1 \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

And the other ($\lambda = 4$):

$$\begin{aligned} (5-4)v_1 - v_2 &= 0 \\ 3v_1 + (1-4)v_2 &= 0 \end{aligned} \Rightarrow v_1 = v_2 \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \Rightarrow \lambda^2 + 2\lambda + 2 = 0 \quad \lambda = -1 + i$

Recall that we only require one eigenvalue/eigenvector when it is complex (for the solution). For practice, you could show that if we use the complex conjugate eigenvalue, we get the complex conjugate eigenvector (but its not necessary here):

$$\begin{aligned} (1 - (-1 + i)v_1 - 5v_2 &= 0 \\ v_1 + (-3 - (-1 + i))v_2 &= 0 \end{aligned} \Rightarrow \begin{aligned} (2 - i)v_1 - 5v_2 &= 0 \\ v_1 + (-2 - i)v_2 &= 0 \end{aligned} \Rightarrow$$

$$\mathbf{v} = \begin{bmatrix} 5 \\ 2 - i \end{bmatrix} \text{ or } \begin{bmatrix} 2 + i \\ 1 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$ Here, $\lambda = 0, 0$

$$v_1 = -3v_2 \Rightarrow \mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Solve for the generalized eigenvector \mathbf{w} :

$$\begin{aligned} 3w_1 + 9w_2 &= -3 \\ -w_1 - 3w_2 &= 1 \end{aligned} \Rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Recall that any choice of w_1, w_2 will work for the solution to the system of differential equations, so we try to choose a "nice" one.

(d) $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \lambda^2 + 4\lambda + 5 = 0$ Complex (we only need one):
 $\lambda = -2 + i$

Compute the eigenvector:

$$\begin{aligned} (-3 + 2 - i)v_1 + 2v_2 &= 0 \\ -v_1 + (-1 + 2 - i)v_2 &= 0 \end{aligned} \Rightarrow \mathbf{v} = \begin{bmatrix} 2 \\ 1 + i \end{bmatrix} \text{ or } \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$$

5. For each system below, find y as a function of x by first writing the differential equation as dy/dx .

(Looks like they are all separable)

(a)

$$\begin{aligned} x' &= -2x \\ y' &= y \end{aligned} \Rightarrow \frac{dy}{dx} = -\frac{y}{2x}$$

$$\frac{1}{y} dy = -\frac{1}{2} \cdot \frac{1}{x} dx \Rightarrow \ln|y| = -\frac{1}{2} \ln|x| + C \Rightarrow y = \frac{A}{\sqrt{x}}$$

(b)

$$\begin{aligned} x' &= y + x^3 y \\ y' &= x^2 \end{aligned} \Rightarrow \frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$$

$$y dy = \frac{x^2}{1+x^3} dx \Rightarrow \frac{1}{2} y^2 = \ln|1+x^3| + C$$

(c)

$$\begin{aligned} x' &= -(2x+3) \\ y' &= 2y-2 \end{aligned} \Rightarrow \frac{dy}{dx} = \frac{-2(y-1)}{2x+3}$$

$$\frac{1}{y-1} dy = \frac{-2}{2x+3} dx \Rightarrow \ln|y-1| = \ln\left(\frac{1}{2x+3}\right) + C \Rightarrow$$

$$y = \frac{A}{2x+3} + 1$$

(d)

$$\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned} \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow y dy = -x dx$$

$$\frac{1}{2} y^2 = -\frac{1}{2} x^2 + C_1 \Rightarrow x^2 + y^2 = C_2$$

6. For matching the graph to the system of equation, the idea was to try putting a vector into the system. For example, consider the point (or vector $[0; 1]^T$). Then, for each system:

(a) $\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We see that, in the lower right graph, at the point $(0, 1)$, the direction of travel is left and up (or $(-1, 1)^T$).

(b) $\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$

We see that, in the upper left graph, at the point $(0, 1)$, the direction of travel is backwards and down, or probably $(-5, -3)^T$.

(c) $\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

In this case, we have a line of fixed points (the two equations are the same), so this must be in the lower left graph.

(d) $\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

This leaves the upper right graph for this one, which also matches our numerical computation at the point $(0, 1)$.