Homework Solutions

1. Give the solution to the four systems $\mathbf{x}' = A\mathbf{x}$ from the previous page (page 5).

System	Solution
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$	$C_1 e^t \begin{bmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(2t) \\ -\cos(2t)+\sin(2t) \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$	$C_1 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$	$C_1 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$	$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left[t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right]$

2. For the three systems below $\mathbf{x}' = A\mathbf{x}$, try to give a rough sketch of the behavior of the solutions near the equilibrium (the origin): Use the Poincare diagram to find:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \mathit{Sink}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow \mathit{DegenerateSource}$$

$$A = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \Rightarrow \mathit{SpiralSink}$$

Note: These problems were here to help you think about the phase plane $((x_1, x_2)$, and not to get an accurate sketch- We would typically use graphing technology for that.

3. For the systems below, use the Poincaré diagram to classify the equilibrium solution. If necessary, first convert the equation to a system of first order equations:

   (a) $\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{x}$ Source

   (b) $y'' + y' + 3y = 0$ Spiral Sink

   (c) $2y'' - 3y' + 4y = 0$ Saddle

   (d) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$ Degenerate source

4. For the following *nonlinear* systems, find the equilibrium solutions (the derivatives are with respect to t, as usual).

   (a) $x' = x - xy$, $y' = y + 2xy$

   $$x(1-y) = 0 \Rightarrow x = 0 \text{ or } y = 1$$

   If $x = 0$, then $y = 0$. If $y = 1$, then $x = -1/2$. Therefore, $(0,0)$ and $(-1/2, 0)$ are the two equilibria.

   (b) $x' = y(2 - x - y)$, $y' = -x - y - 2xy$
   From the first equation, $y = 0$ or $x = 2 - y$. If $y = 0$, then $x = 0$. If $x = 2 - y$ from the first, then $-(2-y) - y - 2(2-y)y = 0$, which simplifies to:

   $$y^2 - 2y - 1 = 0 \Rightarrow y = 1 \pm \sqrt{2}$$

5. Explain how the classification of the origin changes by changing the α in the system:

   $$\mathbf{x}' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} \mathbf{x}$$

   Side note: For organizing your thinking, these points can be plotted on the Poincare Diagram as a vertical line at $\mathrm{Tr}(A) = -2$.

   $$\mathrm{Tr}(A) = -2 \quad \det(A) = -\alpha \quad \Delta = 4 + 4\alpha$$

   Going along the number line:

   - If $\alpha > 0$, the determinant is negative and the origin is a SADDLE.
   - If $\alpha = 0$, there is a line of stable fixed points.
   - If $-1 < \alpha < 0$, the origin is a SINK.
   - If $\alpha = -1$, the origin is a DEGENERATE SINK.
   - If $\alpha < -1$, the origin is a SPIRAL SINK.