Day 4 Starts Here

Today's lecture covers elements from sections 7.6 and 7.8 from our textbook. We will finish computing the solution to $\mathbf{x}' = A\mathbf{x}$ by looking at the case of *complex eigenvalues* and *one real eigenvalue*.

Last time, we saw that, to compute eigenvalues and eigenvectors for a matrix A, we first compute the characteristic equation, then solve for a representative eigenvector.

We applied this to $\mathbf{x}' = A\mathbf{x}$ in the case that we had two distinct real eigenvalues, $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$, and saw that the general solution is:

$$\mathbf{x} = C_1 \lambda_1 t \mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

Case 2: Complex Eigenvalues

First, let's look at the eigenvalue/eigenvector computations themselves in an example: Find the eigenvalues and eigenvectors for the matrix below:

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

SOLUTION: Form the characteristic equation:

$$|A - \lambda I| = 0 \qquad \lambda^2 - Tr(A)\lambda + \det(A) = 0 \qquad \lambda^2 - 4\lambda + 5 = 0 \qquad \lambda = 2 \pm i$$

Now, if $\lambda = 2 + i$, solve for an eigenvector:

$$\begin{array}{rrl} (3-(2+i))v_1 & +v_2 & = 0 \\ v_1 & +(1-(2+i))v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (1-i)v_1 & +v_2 & = 0 \\ v_1 & +(-1-i)v_2 & = 0 \end{array}$$

Recall that we said that these equations needed to be the same line- Indeed they are. To see this, if you divide the first equation by $1 - i$, we get:

$$\frac{1-i}{1-i}v_1 - \frac{2}{1-i}v_2 = 0 \quad \Rightarrow \quad v_1 - \frac{2(1+i)}{(1^2+1^2)}v_2 = 0 \quad \Rightarrow$$

So, if we use the second equation, our eigenvector is any vector that satisfies the equation:

$$v_1 - (1+i)v_2 = 0$$

If v_2 is a free variable, we can write the solution set as:

$$\begin{array}{ll} v_1 & = (1+i)v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

We do not need to re-solve to find the other eigenvector. It is the complex conjugate of the first. That is, we can verify that:

$$\lambda_2 = 2 - i \qquad \mathbf{v}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

(if we have time): Compute $A\mathbf{v}$:

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 3-3i+2 \\ 1-i+1 \end{bmatrix} = \begin{bmatrix} 1-3i \\ 2-i \end{bmatrix}$$

And compare to $\lambda\mathbf{v}$:

$$(2-i) \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 2-2i-i+i^2 \\ 2-i \end{bmatrix} = \begin{bmatrix} 1-3i \\ 2-i \end{bmatrix}$$

Applying Complex evals to Systems of DEs

Suppose we have a complex eigenvalue, $\lambda = a \pm ib$. Use one of them to construct the corresponding eigenvector (complex) $\mathbf{v}$.

Theorem: Given $\lambda = a + ib$, $\mathbf{v}$, the solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

Notice that this is the extension of what we did in Chapter 3.

Example

Give the general solution to the system $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{x}$

This is the system for which we already have the eigenvalues and eigenvectors:

$$\lambda = 2 + i \qquad \mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Now, compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(2+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \mathrm{e}^{2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} =$$

$$\mathrm{e}^{2t} \begin{bmatrix} (\cos(t) - \sin(t)) + i(\cos(t) + \sin(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

so that the general solution is given by:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_1 \mathrm{e}^{2t} \begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{bmatrix}$$

Geometrically, the origin is a *spiral source*. As a side remark, if I had solved the second equation for x_1 and substituted it into the first, I would have had:

$$x_2'' - 4x_2' + 5x_2 = 0 \quad \Rightarrow \quad r = 2 \pm i \quad \Rightarrow \quad x_2 = C_1 \mathrm{e}^{2t}\cos(t) + C_2 \mathrm{e}^{2t}\sin(t)$$

Example

Give the general solution to the system: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

First, the characteristic equation: $\lambda^2 + 1 = 0$, so that $\lambda = \pm i$.

Now we solve for the eigenvector to $\lambda = i$:

$$\begin{aligned} (2-i)v_1 - 5v_2 &= 0 \\ 1v_1 + (-2-i)v_2 &= 0 \end{aligned}$$

Using the second equation, $v_1 = (2+i)v_2$, and we have our eigenvalue/eigenvector pair. Now we compute the needed quantity, $e^{\lambda t}\mathbf{v}$:

$$e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t)) \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos(t) + i\sin(t))(2+i) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

Simplifying, we get:

$$\begin{bmatrix} (2\cos(t) - \sin(t)) + i(2\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

The solution is:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

We will quickly verify that this is what we would get using the techniques of Chapter 3. From the second equation, solve for x_1, then use the first equation to get a second order DE for x_2.

$$x_1 = x_2' + 2x_2 \quad \Rightarrow \quad (x_2'' + 2x_2') = 2(x_2' + 2x_2) - 5x_2 \quad \Rightarrow \quad x_2'' + x_2 = 0$$

Therefore, $x_2 = C_1\cos(t) + C_2\sin(t)$. Solving for x_1:

$$x_1 = x_2' + 2x_2 = (-C_1\sin(t) + C_2\cos(t)) + 2(C_1\cos(t) + C_2\sin(t))$$

and we see that we get the identical solution.

Graphically, the solutions are ellipses. In fact, if we solve the differential equation by computing dy/dx, we get solutions of the form:

$$x^2 - 4xy + 5y^2 = C$$

Graphical Summary- Complex Eigenvalues

Notice that if the real part of λ is positive, solutions "blow up". If the real part of λ is negative, $y(t) \to 0$ as $t \to \infty$. Therefore, the origin can be classified by $\lambda = \alpha \pm \beta i$:

- If $\alpha = 0$, we get pure periodic solutions (the period depends on β).
- If $\alpha < 0$, the origin is a *spiral sink.*
- If $\alpha > 0$, the origin is a *spiral source.*

Case 3: One Real Eigenvalue, One Eigenvector

In the rare occurrence that you have one eigenvalue but two eigenvectors (we'll do this in class), go to Case 1- For example, find the eigenvalues and eigenvectors to the identity matrix.

$$\begin{vmatrix} (1-\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

Now, solve the system for $\mathbf{v}$:

$$\begin{aligned} 0v_1 + 0v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned}$$

Both v_1, v_2 are free. Therefore, we can write:

$$\mathbf{v} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And therefore, we have two eigenvectors, $[1,0]^T$ and $[0,1]^T$. This is not typical.

Typical Case: A double eigenvalue, one eigenvector

Example: $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ In this case, $\lambda = 2, 2$ but

$$\begin{aligned} 0v_1 + 3v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

You can read pages 423-424 for more information on this one. This is a special case where we need to find a second eigenvector (called a generalized eigenvector). We will summarize that information here.

Definition: Generalized eigenvector

Given a matrix A, then eigenvector $\mathbf{v}$ is found by solving the system:

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}$$

The "generalized" eigenvector $\mathbf{w}$ is found by solving the system:

$$(A - \lambda I)\mathbf{w} = \mathbf{v} \quad \Rightarrow \quad \begin{aligned} (a-\lambda)w_1 + bw_2 &= v_1 \\ c\, w_1 + (d-\lambda)w_2 &= v_2 \end{aligned}$$

You might notice that $\mathbf{w}$ satisfies the equation $A^2\mathbf{w} = \lambda\mathbf{v}$.

Another motivation for the equation

Given matrix A with eigenvalue λ and eigenvector $\mathbf{v}$, if $e^{\lambda t}(t\mathbf{v}+\mathbf{w})$ is to solve $\mathbf{x}' = A\mathbf{x}$, find what must be true about $\mathbf{w}$.

SOLUTION: First compute $\mathbf{x}'$:

$$\lambda e^{\lambda t}(t\mathbf{v} + \mathbf{w}) + e^{\lambda t}\mathbf{v}$$

Now compute $A\mathbf{x}$, and use the fact that $\mathbf{v}$ is an eigenvec of A:

$$Ae^{\lambda t}(t\mathbf{v} + \mathbf{w}) = e^{\lambda t}(tA\mathbf{v} + A\mathbf{w}) = e^{\lambda t}(\lambda t\mathbf{v} + A\mathbf{w})$$

Put the two statements together:

$$\lambda e^{\lambda t}(t\mathbf{v} + \mathbf{w}) + e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda t\mathbf{v} + A\mathbf{w})$$

$$e^{\lambda t}\left(\lambda\mathbf{w} + \mathbf{v} = A\mathbf{w}\right)$$

or

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

Solving the DE with a Single Eval

Suppose that the matrix A has a double eigenvalue with only one corresponding eigenvector $\mathbf{v}$. Then the solution to the differential equation is given by:

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}\left(t\mathbf{v} + \mathbf{w}\right)$$

Of course, in this instance we can always use the method of Chapter 3 to solve this, but we want to note the form of the solution before we talk about the geometry in Chapter 9.

Example:

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\mathbf{x}$$

The trace is 0 and the determinant is 0. Therefore, $\lambda = 0$ is the only eigenvalue. We now get the eigenvector $\mathbf{v}$:

$$4v_1 - 2v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now the generalized eigenvector $\mathbf{w}$:

$$\begin{array}{rl} 4w_1 - 2w_2 & = 2 \\ 8w_1 - 4w_2 & = 4 \end{array} \qquad 4w_1 - 2w_2 = 2$$

We take any w_1, w_2 that satisfies this relationship- integer solutions are nice (you can change $\mathbf{v}$ if necessary), and in this case we choose $w_1 = 0$ and $w_2 = -1$.

The solution is (in several forms):

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_2 \left(t \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2c_1 + 2c_2 t \\ (4c_1 - c_2) + 4tc_2 \end{bmatrix}$$

We'll check that this is indeed a solution. First, we compute $\mathbf{x}'$ and show that it is equal to $A\mathbf{x}$:

$$\mathbf{x}' = \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} 2c_1 + 2c_2 t \\ (4c_1 - c_2) + 4tc_2 \end{bmatrix} = \begin{bmatrix} 0 + 2c_2 + 0t \\ 0 + 4c_2 + 0t \end{bmatrix}$$

Summary

To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$. Get $\mathbf{w}$ that solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1 \mathbf{v} + C_2 \left(t\mathbf{v} + \mathbf{w}\right)\right)$$