**Solutions to Selected Problems, 3.7 (Model of Mass-Spring System)**

**NOTE about units:** On quizzes/exams, we will always use the standard units of meters, kilograms and seconds, or feet, pounds and seconds. The textbook likes to mix them up somewhat.

5. A mass weighing 2 lb stretches a spring 6 inches.

   *Remark:* This information is here so that we can get the spring constant. Change the 6 inches to 1/2 foot:

$$mg - kL = 0 \quad \Rightarrow 2 - \frac{k}{2} = 0 \quad \Rightarrow \quad k = 4$$

   From this, we can also get the mass using $g = 32$ ft/sec$^2$ (the constant would be given to you):

$$mg = 2 \quad \Rightarrow \quad m = \frac{2}{g} = \frac{2}{32} = \frac{1}{16}$$

   Continuing with the problem, we only need to determine $\gamma$- Since there is no damping, $\gamma = 0$, and

$$\frac{1}{16}u'' + 0u' + 4u = 0 \quad \Rightarrow \quad u'' + 64u = 0$$

   If the mass is pulled down 3 inches and released, the initial conditions are $u(0) = \frac{1}{4}$ and $u'(0) = 0$. Solving the IVP, we get

$$u(t) = \frac{1}{4}\cos(8t)$$

   so the amplitude is 1/4 and the period is $2\pi/8$ or $\pi/4$. You don't need to plot it for now.

7. Very similar to 5.- For the exam, I will not ask you to determine the frequency, period amplitude, phase of the motion (but be able to solve the IVP).

9. This one is tricky in terms of the units (See the note at the top), but if we continue, we will write cm rather than meters, and 9.8 becomes 980. The spring constant:

$$k = \frac{20 \cdot 980}{5} = 3920 \text{ dyne/cm}$$

   And the IVP:

$$20u'' + 400u' + 3920u = 0 \qquad u(0) = 2, u'(0) = 0$$

   (where $u$ is measured in cm and time is in seconds).

11. (Watch the units!) Building the model, the spring constant is

$$k = \frac{3}{0.1} = 30 \text{ N/m}$$

and the damping coefficient:

$$\gamma u' = F_d \quad \Leftrightarrow \gamma(5) = 3 \quad \Rightarrow \gamma = \frac{3}{5}$$

so that

$$2u'' + \frac{3}{5}u' + 30u = 0$$

(The numbers are a little messy in the solution).

24. We'll work this out later. Mainly,

1. Problem 6: *Note* that there may be some confusion over which units to use.

   The constants in this problem (the spring constant is found using the relation: $mg - kL = 0$, or $k = mg/L$):

   - Mass: 100 grams or 0.1 kg
   - Gravity: 9.8 meters/second$^2$
   - Length: 5 cm or 0.05 meters
   - Spring constant: $k = (9.8)(0.1)/0.05 = 19.6$
   - The initial velocity: 10 cm or 0.1 meters

   So, using meters, kg, seconds:

   $$0.1u'' + 19.6u = 0 \text{ or } u'' + 196u = 0 \qquad r = 14$$

   $$u = C_1 \cos(14t) + C_2 \sin(14t) \qquad u(0) = 0 \quad u'(0) = 0.1$$

   so that

   $$u = \frac{1}{140} \sin(14t) \text{ in meters, or } \quad \frac{5}{7} \sin(14t) \text{ cm}$$

   In either case, the time (in seconds) to equilibrium:

   $$\sin(14t) = 0 \quad \Rightarrow \quad 14t = 0, \pi, 2\pi, \ldots$$

   We want the first time we return to equilibrium, so $14t = \pi$, or $t = \pi/14$.

2. Problem 12: From the discussion on P. 201-202, we have:

   $$LQ'' + RQ' + \frac{1}{C}Q = 0$$

with $L = 2 \times 10^{-1}$, $R = 3 \times 10^2$, and $1/C = 1/10^{-5} = 10^5$. Therefore,

$$2 \times 10^{-1} Q'' + 3 \times 10^2 Q' + 10^5 Q = 0 \text{ or } 2Q'' + 3 \times 10^3 Q' + 10^6 Q = 0$$

From the characteristic equation:

$$r = \frac{-3 \times 10^3 \pm \sqrt{9 \times 10^6 - 8 \times 10^6}}{4} = 10^3 \left( \frac{-3 \pm 1}{4} \right) = -500, -1000$$

Therefore,

$$Q = C_1 \mathrm{e}^{-500t} + C_2 \mathrm{e}^{-1000t}$$

The initial conditions translate to: $Q(0) = 10^{-6}$ and $Q'(0) = 0$. give $C_1 = 2 \times 10^{-6}$ and $C_2 = -10^{-6}$, or

$$Q = 10^{-6} \left( 2\mathrm{e}^{-500t} - \mathrm{e}^{-1000t} \right)$$

3. Problem 13: The quasi-period of the damped motion is 50% greater than the period of the undamped motion, so we need to find the period of the undamped motion:

$$u'' + u = 0 \quad \Rightarrow \quad u = A\cos(t) + B\sin(t)$$

The period is $2\pi/1$, or $2\pi$. The quasi-period of the damped motion is then $3\pi$. If we assume a form of $R\cos(\omega t - \delta)$, then:

$$\frac{2\pi}{\omega} = 3\pi \quad \Rightarrow \quad \omega = \frac{2}{3}$$

The roots to the characteristic equation of the damped motion are:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

If we want to get quasi-periodic motion, then $\gamma^2 - 4 < 0$, so we'll write $\sqrt{\gamma^2 - 4}$ as $i\sqrt{4 - \gamma^2}$, and:

$$r = -\frac{\gamma}{2} \pm \frac{\sqrt{4 - \gamma^2}}{2} = \lambda \pm \omega i$$

We want $\omega = \frac{2}{3}$, so:

$$\frac{\sqrt{4 - \gamma^2}}{2} = \frac{2}{3} \quad \Rightarrow \quad \gamma^2 = 4 - \frac{16}{9} = \frac{20}{9}$$

so $\gamma = \frac{2\sqrt{5}}{3}$.

4. Problem 14: The period of motion from an undamped system is from the solution to:

$$mu'' + ku = 0 \quad \Rightarrow \quad u = R\cos\left( \sqrt{\frac{k}{m}} t - \delta \right)$$

The period is:

$$\frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

Recall that $mg - kL = 0$, or $k = mg/L$. Replacing $k$ in the above equation gives us the result,

$$2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mL}{mg}} = 2\pi\sqrt{\frac{L}{g}}$$

5. Problem 15: By the linearity of the differential equation, if $v, w$ each solves $mu'' + \gamma u' + ku = 0$, then so does $v + w$ (in fact, any linear combination $c_1 v + c_2 w$ solves it).

   Therefore, we just need to check that $u = v + w$ satisfies the initial conditions:

$$u(t_0) = v(t_0) + w(t_0) = u_0 + 0 = u_0$$

$$u'(t_0) = v'(t_0) + w'(t_0) = 0 + u_0' = u_0'$$

6. Problem 24: Consider

$$\frac{3}{2}u'' + ku = 0 \qquad u(0) = 2 \quad u'(0) = v$$

   We want to determine $k, v$ so that the amplitude of our solution is 3 and the period is $\pi$. First get the solution by solving the characteristic equation:

$$\frac{3}{2}r^2 + k = 0 \quad \Rightarrow \quad r = \pm\sqrt{\frac{2k}{3}}$$

   Therefore, the period of our solution will be:

$$\text{Period} = \frac{2\pi}{\sqrt{\frac{2k}{3}}} = \pi$$

   Solve this for $k$ to get that $k = 6$, which means that $r = \pm 2i$.

   Our solution is now:

$$u = C_1 \cos(2t) + C_2 \sin(2t)$$

   Using the initial conditions, $C_1 = 2$ and $C_2 = \frac{v}{2}$

   The amplitude can be determined now in terms of $v$:

$$R^2 = C_1^2 + C_2^2 \quad \Rightarrow \quad 9 = 4 + \frac{v^2}{4}$$

   we get that $v^2 = 20$, or $v = \pm 2\sqrt{5}$.

7. Problem 26: Probably best to leave in general constants until the very end.

   The solutions to the characteristic equation are:

$$r = -\frac{\gamma}{2m} \pm \frac{\sqrt{4km - \gamma^2}}{2m} i \doteq \lambda \pm \mu i$$

   so that the general solution is:

$$u = e^{\lambda t} \left( c_1 \cos(\mu t) + c_2 \sin(\mu t) \right)$$

   Solve using the initial conditions $u(0) = u_0, u'(0) = v_0$, we get:

$$c_1 = u_0 \qquad c_2 = \frac{v_0 - \lambda u_0}{\mu}$$

   so that the (squared) amplitude is:

$$R^2 = c_1^2 + c_2^2 = u_0^2 + \left( \frac{v_0 - \lambda u_0}{\mu} \right)^2$$

   We substitute the values in for $\lambda$ and $\mu$ noticing that

$$\frac{1}{\mu^2} = \frac{4m^2}{4km - \gamma^2}$$

   Now:

$$R^2 = u_0^2 + \frac{(v_0 + \frac{\gamma}{2m} u_0)^2 \cdot 4m^2}{4km - \gamma^2} = u_0^2 + \frac{(2mv_0 + \gamma u_0)^2}{4km - \gamma^2}$$

   This answer is equivalent to the text's answer, but I think it's easier to read.

   In any event, it is clear that, as $\gamma \to 4km$, the amplitude increases (the period increases as well).