Selected Solutions, Section 6.5

2. Solve $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$, $y(0) = 0$ and $y'(0) = 0$.

   After taking the Laplace transform, we solve for $Y(s)$:

$$Y = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s^2+4}$$

   Think of this as:

$$\left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right) H(s)$$

   so that once we find $h(t)$, the inverse transform will be:

$$u_\pi(t)h(t-\pi) - u_{2\pi}(t)h(t-2\pi)$$

   In this case,

$$H(s) = \frac{1}{s^2+4} \quad \Rightarrow \quad h(t) = \frac{1}{2}\sin(2t)$$

   The solution to the IVP can be simplified since h is periodic with period π:

$$y(t) = u_\pi(t)h(t-\pi) - u_{2\pi}(t)h(t-2\pi) = h(t)(u_\pi(t) - u_{2\pi}(t))$$

   Therefore, the solution can be written as:

$$y(t) = \begin{cases} \frac{1}{2}\sin(2t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

3. $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$, $y(0) = 0$, $y'(0) = 1/4$.

   Sometimes it is useful to think about what the ODE is before solving it. In this case,

   > If this represented the model of a mass-spring system, notice that there are three distinct phases of motion- The first begins at time 0, when we have no forcing, but an initial velocity of 1/2. If left alone, the homogeneous solution would die off quickly. However, at $t = 5$, the system is given a unit impulse, which starts the system off again (although with a larger velocity that the initial velocity). Again, if left alone, the system would quickly go back to equilibrium. Finally, at time 10, we start a constant force of 1, and continue that through time- We expect our solution to become constant as well (since the homogeneous part of the solution will die off).

   Now we'll solve it algebraically and plot the result in Wolfram Alpha:

$$(s^2+3s+2)Y = \mathrm{e}^{-5s} + \frac{\mathrm{e}^{-10s}}{s} + \frac{1}{2}$$

$$Y(s) = \frac{1}{s^2+3s+2}\left(\mathrm{e}^{-5s} + \frac{1}{2}\right) + \mathrm{e}^{-10s}\frac{1}{s(s^2+3s+2)}$$

   We have two sets of partial fractions to compute:

$$\frac{1}{s^2+3s+2} = -\frac{1}{s+2} + \frac{1}{s+1}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1}$$

Therefore, the inverse Laplace transform gives:

$$y(t) = \frac{1}{2}\left(-\mathrm{e}^{-2t} + \mathrm{e}^{-t}\right) + u_5(t)\left(\mathrm{e}^{-(t-5)} - \mathrm{e}^{-2(t-5)}\right) + u_{10}(t)\left(\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2(t-10)} - \mathrm{e}^{-(t-10)}\right)$$

The solution corresponds to what we had expected. In Wolfram Alpha, we can plot the solution:

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solve  y''+3y'+2y=Dirac(t-5)+Heaviside(t-10), with y(0)=0, y'(0)=1/2
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5. The partial fractions here are a little heavy. Here's how I might do them:

$$\frac{1}{(s^2+1)(s^2+2s+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+3}$$

so that

$$1 = (As+B)(s^2+2s+3) + (Cs+D)(s^2+1)$$

This leads us to the system of equations:

$$\begin{array}{c|ll} s^3 \text{ terms} & 0 & = A + C \\ s^2 \text{ terms} & 0 & = B + 2A + D \\ s \text{ terms} & 0 & = 3A + 2B + C \\ \text{Constants} & 1 & = 3B + D \end{array}$$

Using the second and fourth equations, we might get a nice substitution:

$$\begin{array}{rl} -B - D & = 2A \\ 3B + D & = 1 \\ \hline 2B & = 1 + 2A \end{array} \qquad \Rightarrow \qquad B = \frac{1}{2} + A$$

And with Equation 1, $B = \frac{1}{2} - C$. Put these into Equation 3 and we can solve for C:

$$0 = 3(-C) + 2(1/2 - C) + C \qquad \Rightarrow \qquad C = \frac{1}{4}$$

From which we now have $A = -1/4$, $B = 1/4$, $C = 1/4$ and $D = 1/4$.

7. The tricky part here is computing the Laplace transform of $\delta(t-c)f(t)$:

$$\mathcal{L}(\delta(t-c)f(t)) = \int_0^\infty \mathrm{e}^{-st}\delta(t-c)f(t)\,dt = \mathrm{e}^{-sc}f(c)$$

where we note that $f(c)$ is a constant (evaluate f at c). In this particular case,

$$\mathcal{L}(\delta(t-2\pi)\cos(t)) = \mathrm{e}^{-2\pi s}\cdot 1$$

14. It may be easiest to do this generally, then look at what happens for specific values of γ:

$$y'' + \gamma y' + y = \delta(t-1) \qquad y(0) = y'(0) = 0$$

Take the Laplace transform of both sides and solve for $Y(s)$:

$$Y = \frac{\mathrm{e}^{-s}}{s^2 + \gamma s + 1}$$

Our choice of table entry for inversion depends on whether or not the denominator is irreducible. We can tell by completing the square:

$$s^2 + \gamma s + 1 = \left(s + \frac{\gamma}{2}\right) + \left(1 - \frac{\gamma^2}{4}\right) = \left(s + \frac{\gamma}{2}\right)^2 + \left(\frac{\sqrt{4-\gamma^2}}{2}\right)^2$$

In the cases we are asked to consider, $\gamma = 1/2, 1/4$ and 0, the denominator is irreducible. Now invert the transform: Given

$$H(s) = \frac{2}{\sqrt{4-\gamma^2}} \frac{\frac{\sqrt{4-\gamma^2}}{2}}{\left(s+\frac{\gamma}{2}\right)^2 + \left(\frac{\sqrt{4-\gamma^2}}{2}\right)^2}$$

then

$$h(t) = \frac{2}{\sqrt{4-\gamma^2}} \mathrm{e}^{-(\gamma/2)t} \sin\left(\frac{\sqrt{4-\gamma^2}}{2}\, t\right)$$

The overall solution is then $u_1(t)h(t-1)$.

For parts (b) and (c), we are meant to use the computer to solve for the maximum. We can answer part (d): If $\gamma = 0$, then solution simplifies to

$$h(t) = \sin(t)$$

so that the maximum of $h(t)$ occurs at $t = \pi/2$ (so the maximum of $h(t-1)$ occurs at $t = 1 + \pi/2$.

15. The solution to this one is almost identical to the previous problem, except we multiply by k:

$$y(t) = k u_1(t) h(t-1)$$

where $h(t)$ was found in #14. The remaining problems are meant to be done on a computer.

16. Omit this problem.

17-19. In these problems, we work with using the sum. Try to think about how the sum of the impulses will effect your solution- The homogeneous solution is simply a sum of $\sin(t)$ and $\cos(t)$, so that the homogeneous part of the solution has a period of 2π.

17. In this problem, the first impulse occurs at $t = \pi$, and so that will start a sine function:

$$(s^1 + 1)Y(s) = e^{-\pi s} \quad \Rightarrow \quad Y(s) = \frac{e^{-\pi s}}{s^2+1} \quad \Rightarrow \quad y(t) = u_\pi(t)\sin(t-\pi)$$

At $t = 2\pi$ comes our next unit impulse. Note (from a sketch of $y(t)$) that $y'(2\pi) = -1$, so the impulses will cancel each other out. Algebraically,

$$y(t) = u_\pi(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi) = u_{\pi(t)}(-\sin(t)) + u_{2\pi}(t)\sin(t)$$

Writing it piecewise,

$$y(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ -\sin(t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

Now when $\delta(t-3\pi)$ comes along, it starts the same motion as before (then $\delta(t-4\pi)$ turns it off again, then $\delta(t-5\pi)$ starts it up again, etc.). Therefore, the solution (in piecewise form) is the following- After time 20π, the solution will be zero following the pattern:

$$y(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ -\sin(t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi \le t < 3\pi \\ -\sin(t) & \text{if } 3\pi \le t < 4\pi \\ \vdots & \vdots \\ -\sin(t) & \text{if } 19\pi \le t < 20\pi \\ 0 & \text{if } t \ge 20\pi \end{cases}$$

18. In Exercise 18, we start the same way- at $t = \pi$ we impart a unit impulse, and that starts a sine function going (that is, $\sin(t-\pi) = -\sin(t)$).

After π units of time ($t = 2\pi$), the curve has a velocity of -1, and we impart an additional unit impulse in the negative direction (that will make the amplitude increase by 1).

Similarly, at $t = 3\pi$, the curve now has a velocity of 2, and we will impart an additional unit impulse in the positive direction (so the amplitude increases to 3).

The same thing happens at $t = 4\pi$, $t = 5\pi$, etc. Therefore, the sine function will continue to grow 1 unit in amplitude for every π units in time until we get to 20π. After that, the solution will have an amplitude of 20.

Let's see if we can show that algebraically: We know that

$$\sin(t - k\pi) = -\sin(t) \qquad k = 1, 3, 5, 7, \cdots$$

and

$$\sin(t - k\pi) = \sin(t) \qquad k = 2, 4, 6, 8, \cdots$$

Therefore,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} u_{k\pi}(t) \sin(t - k\pi) = \begin{cases} 0 & \text{if } 0 \le t \le \pi \\ -\sin(t) & \text{if } \pi \le t < 2\pi \\ -2\sin(t) & \text{if } 2\pi \le t < 3\pi \\ -3\sin(t) & \text{if } 3\pi \le t < 4\pi \\ \vdots & \vdots \\ -19\sin(t) & \text{if } 19\pi \le t < 20\pi \\ -20\sin(t) & \text{if } t \ge 20\pi \end{cases}$$

19. This one is more complex since the "hits" don't occur at the end of a period (rather they occur in the middle of a period).

    We can analyze this easiest by writing the solution piecewise using the following substitutions (do them graphically if you're not sure):

$$\begin{aligned} \sin(t - \pi/2) &= \cos(t) \\ \sin(t - \pi) &= -\sin(t) \\ \sin(t - 3\pi/2) &= -\cos(t) \\ \sin(t - 2\pi) &= \sin(t) \vdots \end{aligned}$$

    Therefore, we end up with a function that is 2π periodic:

$$y(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi/2 \\ \cos(t) & \text{if } \pi/2 \le t < \pi \\ \cos(t) - \sin(t) & \text{if } \pi \le t < 3\pi/2 \\ -\sin(t) & \text{if } 3\pi/2 \le t < 2\pi \\ 0 & \text{if } 2\pi \le t < 3\pi/2 \\ \vdots & \vdots \\ 0 & \text{if } t \ge 20\pi \end{cases}$$