

## Solutions to Extra Practice Problems

### Linear Operators and Cramer's Rule

1. Let  $R(f)$  be the operator defined by:  $R(f) = f''(t) + 3t^2 f(t)$ . Find  $R(f)$  for each function below:

(a)  $f(t) = t^2$ : Since  $f'(t) = 2t$  and  $f''(t) = 2$ , we have:

$$R(t^2) = 2 + 3t^2 \cdot t^2 = 2 + 3t^4$$

(b)  $f(t) = \sin(3t)$ : Since  $f'(t) = 3 \cos(3t)$  and  $f''(t) = -9 \sin(3t)$ , we substitute:

$$R(\sin(3t)) = -9 \sin(3t) + 3t^2 \sin(3t) = (3t^2 - 9) \sin(3t)$$

(c)  $f(t) = 2t - 5$

$$R(2t - 5) = 0 + 3t^2(t - 5) = 3t^3 - 15t^2$$

2. Let  $R$  be the operator defined in the previous problem. Show that  $R$  is a linear operator.

- $R(f + g) = (f + g)'' + 3t^2(f + g) = f'' + g'' + 3t^2 f + 3t^2 g = f'' + 3t^2 f + g'' + 3t^2 g = R(f) + R(g)$
- $R(cf) = (cf)'' + 3t^2(cf) = cf'' + 3t^2 cf = c(f'' + 3t^2 f) = cR(f)$

3. Let  $F(y) = y'' + y - 5$ . Explain why  $F$  is not linear.

Best way is to show it- This  $F$  does not satisfy either part of the definition. For example,

$$F(x + y) = (x + y)'' + (x + y) - 5 = x'' + x + y'' + y - 5 \neq (x'' + x - 5) + (y'' + y - 5) = F(x) + F(y)$$

4. Find the operator associated with the given differential equation, and classify it as linear or not linear:

(a)  $y' = ty^2 + \cos(t)$

$$L(y) = y' - ty^2$$

Not linear

(b)  $y'' = 4y' + 3y + \sin(t)$

$$L(y) = y'' - 4y' - 3y$$

Linear

(c)  $y' = e^t y + 5$

$$L(y) = y' - e^t y$$

Linear (in  $y$ )

(d)  $y'' = -\cos(y) + \cos(t)$

$$L(y) = y'' + \cos(y)$$

Not linear

5. Use Cramer's Rule to solve the following systems:

(a) 
$$\begin{array}{rcl} C_1 + C_2 & = & 2 \\ -2C_1 - 3C_2 & = & 3 \end{array} \quad \text{SOLUTION:}$$

$$C_1 = \frac{\begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{-9}{-1} = 9 \quad C_2 = \frac{\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{7}{-1} = -7$$

$$(b) \quad \begin{array}{rcl} C_1 + C_2 & = & y_0 \\ r_1 C_1 + r_2 C_2 & = & v_0 \end{array}$$

$$C_1 = \frac{r_2 y_0 - v_0}{r_2 - r_1} \quad C_2 = \frac{v_0 - r_1 y_0}{r_2 - r_1}$$

$$(c) \quad \begin{array}{rcl} C_1 + C_2 & = & 2 \\ 3C_1 + C_2 & = & 1 \end{array}$$

$$C_1 = -\frac{1}{2} \quad C_2 = \frac{5}{2}$$

$$(d) \quad \begin{array}{rcl} 2C_1 - 5C_2 & = & 3 \\ 6C_1 - 15C_2 & = & 10 \end{array} \quad \text{The denominator to Cramer's Rule is zero- No solution.}$$

$$(e) \quad \begin{array}{rcl} 2x - 3y & = & 1 \\ 3x - 2y & = & 1 \end{array}$$

SOLUTION:

$$x = \frac{1}{5}, y = \frac{-1}{5}$$

6. Suppose  $L$  is a linear operator. Let  $y_1, y_2$  each solve the equation  $L(y) = 0$  (so that  $L(y_1) = 0$  and  $L(y_2) = 0$ ). Show that anything of the form  $c_1 y_1 + c_2 y_2$  will also solve  $L(y) = 0$ .

SOLUTION:

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

*Go back to Section 2.4, page 76 and look at Exercises 23-26. This section generalizes those results.*