M244
Spring 2016

Quiz 11 Solutions

1. Given the matrix A below, if $\lambda = 2+i$ is an eigenvalue, find an eigenvector: $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

   SOLUTION: Putting down both equations:

$$\begin{array}{rrl} (1-(2+i))v_1 & -2v_2 & = 0 \\ v_1 & +(3-(2+i))v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} -(1+i)v_1 & -2v_2 & = 0 \\ v_1 & +(1-i)v_2 & = 0 \end{array}$$

   If you use the first equation, you might take the following, which can be simplified a bit:

$$\begin{array}{ll} v_1 & = \frac{2}{-(1+i)} v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ -(1+i) \end{bmatrix}$$

   If you use the second equation:

$$\begin{array}{ll} v_1 & = (-1+i)v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

   Either choice is fine.

2. If $\lambda = 1+i$ and $\mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is an eigenvalue and eigenvector for matrix A, write the solution to $\mathbf{x}' = A\mathbf{x}$:

   SOLUTION: We need to recall that $\mathrm{e}^{\lambda t} = \mathrm{e}^{(1+i)t} = \mathrm{e}^t \mathrm{e}^{it} = \mathrm{e}^t(\cos(t) + i\sin(t))$. Now when we multiply:

$$\mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^t(\cos(t) + i\sin(t)) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \mathrm{e}^t \begin{bmatrix} (\cos(t) - \sin(t)) + i(\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

   Now we take $C_1\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2\mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v})$:

$$\mathrm{e}^t \left(C_1 \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) + \cos(t) \\ \sin(t) \end{bmatrix} \right)$$

3. Find the eigenvalues and eigenvectors for the matrix: $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

   SOLUTION: The characteristic equation is $\lambda^2 + 4\lambda + 3 = 0$, so $\lambda = -1$ and $\lambda = -3$. The eigenvector for $\lambda = -1$ can be found by solving:

$$(-2+1)v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Similarly, for $\lambda = -3$:

$$(-2+3)v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = -v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$