Example Solutions, Exam 3, Math 244

1. Consider an unforced harmonic oscillator, and find the general solution if we have two complex roots to the characteristic equation.

   SOLUTION: The equation is $y'' + py' + qy = 0$. The roots to the characteristic equation are:

$$s = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

   If we assume complex roots, we can re-write the expression as:

$$s = -\frac{p}{2} \pm \frac{\sqrt{4q - p^2}}{2}\, i$$

   and the general solution would then be:

$$y(t) = \mathrm{e}^{-pt/2}\left(C_1 \cos\left(\frac{\sqrt{4q-p^2}}{2}\, t\right) + C_2 \sin\left(\frac{\sqrt{4q-p^2}}{2}\, t\right)\right)$$

   (a) For the rest of the questions, assume mass $m = 1$, damping factor $\gamma = 1$ and spring constant $k = 2$, and re-write the solution.

   SOLUTION: Using those values of our constants, we take $p = 1$ and $q = 2$ and we get:

$$y(t) = y(t) = \mathrm{e}^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}\, t\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}\, t\right)\right)$$

   (b) From which point forward will the exponential part of the solution be less than 1/10?

   SOLUTION: $\mathrm{e}^{-t/2} = 1/10$, so $-t/2 = \ln(1/10)$ and $t = -2\ln(1/10) = 2\ln(10)$

   (c) What is the natural (pseudo-)period?

   The period is

$$\frac{4\pi}{\sqrt{7}}$$

   (d) If k is increased slightly, does the natural period increase or decrease? (You might look back at the general case).

   SOLUTION: Since the mass has 1 unit, the k in our general model is the same as q in this model. Increasing q will increase the value $4q - p^2$. Since this quantity is in the denominator of the period, increasing q will decrease the period.

2. Give the solution to the following initial value problems (IVP):

   (a) $u'' + 6u' + 9u = 0$ with $u(0) = 2$ and $u'(0) = 7/3$

   SOLUTION: $s^2 + 6s + 9 = 0$ gives $s = -3, -3$, so the general solution is

$$u(t) = \mathrm{e}^{-3t}(C_1 + C_2 t) \qquad u'(t) = \mathrm{e}^{-3t}(-3C_1 + C_2 - 3C_2 t)$$

   so solving for the coefficients, we get $C_1 = 2$, and

$$-6 + C_2 = \frac{7}{3} \quad \Rightarrow \quad C_2 = \frac{25}{3}$$

   The solution is

$$u(t) = \mathrm{e}^{-3t}\left(2 + \frac{25}{3}t\right)$$

   (b) $u'' + 6u' + 8u = 0$ with $u(0) = 1$ and $u'(0) = 0$

   SOLUTION:

$$u(t) = 2\mathrm{e}^{-2t} - \mathrm{e}^{-4t}$$

(c) $2u'' + 3u = 0$ with $u(0) = 2$ and $u'(0) = -3$.
SOLUTION: $u'' + \frac{3}{2}u = 0$ is our undamped, unforced harmonic oscillator. Using the initial values, we get:

$$u(t) = 2\cos\left(\sqrt{\frac{3}{2}}\,t\right) - \sqrt{6}\sin\left(\sqrt{\frac{3}{2}}\right)$$

NOTE: It would be OK to leave $\sqrt{6}$ as $3\sqrt{\frac{2}{3}}$

(d) $u'' + 4u' + 5u = 0$ with $u(0) = 1$ and $u'(0) = 1$.
SOLUTION:

$$u(t) = \mathrm{e}^{-2t}\cos(t) + 3\mathrm{e}^{-3t}\sin(t)$$

3. Find the general solution:

(a) $y'' + 6y' + 8y = \mathrm{e}^{-t} + 3t^2 + \sin(3t)$
SOLUTION: Break it into 4 pieces:

- $y'' + 6y' + 8y = 0$.
  For this, we get $y_h(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{-4t}$.
- $y'' + 6y' + 8y = \mathrm{e}^{-t}$
  For this, we guess $y_p = A\mathrm{e}^{-t}$. We do not need to multiply by t in this case. Now solve for A:

$$A\mathrm{e}^{-t}(1 - 6 + 8) = \mathrm{e}^{-t} \quad \Rightarrow \quad A = \frac{1}{3}$$

- $y'' + 6y' + 8y = 3t^2$
  Guess $At^2 + Bt + C$, which we substitute into the DE:

$$\begin{array}{rlccc} 8y_p & = 8At^2 & +8Bt & +8C \\ 6y_p' & = & 6(2At) & +6B \\ y_p'' & = & & 2A \\ \hline 3t^2 & = 8At^2 & +(8B+12A)t & +(8C+6B+2A) \end{array} \quad \Rightarrow \quad y_p = \frac{3}{8}t^2 - \frac{9}{16}t + \frac{21}{64}$$

- $y'' + 6y' + 8y = \sin(3t)$
  We can use the complexified form, and take the imaginary part of $A\mathrm{e}^{3it}$:

$$A\mathrm{e}^{3it}(-9 + 6(3i) + 8) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-1+18i} = \frac{-1-18i}{325}$$

  Sorry about the big denominator. I'll try to be careful on the exam so that the numbers don't turn out too badly. In this case, find the imaginary part of $A\mathrm{e}^{3it}$, which is:

$$y_p(t) = -\frac{18}{325}\cos(3t) - \frac{1}{325}\sin(3t)$$

4. (Exercise 23 in Ch 4 review) Eight second order equations and four graphs are given below. For each **graph**, determine the differential equation for which $y(t)$ is a solution, and briefly state how you know your answer is correct. You should do this exercise without any technology.

SOLUTION: Before we go to the graphs, let's make notes about what kind of solution we expect from each DE.

(a) $y'' + 16y = 0$
SOLUTION: This has pure periodic solutions, with period $2\pi/4$, or $\pi/2$. Graph (a) is a possibility, since it is also purely periodic, but it looks like this graph does not have this period. No graph corresponds to this DE.

(b) $y'' + 5y' + 5y = 5\cos(2t)$

SOLUTION: This has a relatively high damping factor, so we expect the homogeneous part of the solution to die off quickly, leaving the particular solution. The period of the particular solution (which is also the steady state solution in this case) will be $2\pi/2 = \pi$. This is consistent so far with graph (b)- It looks like it takes about 6 units to complete 2 cycles, or 3 units per cycle (which is approximately π). We should go ahead and take GRAPH B to be the solution,

(c) $y'' + 5y' + y = 5\cos(4t)$

SOLUTION: This one is similar to the last, but the period of the response (and steady state) is $2\pi/4$, or $\pi/2$. That doesn't match graph (b), which is the only graph that might work.

(d) $2y'' - y' + 10y = 0$

SOLUTION: The next two functions might be consistent with graph (c). Checking the solutions to the characteristic equation:

$$s = \frac{1 \pm \sqrt{1 - 4(2)(10)}}{4}$$

so complex roots with $e^{t/4}$ in front, which would blow up the solutions. No graph would work with this one.

(e) $2y'' + y' + 10y = 0$

SOLUTION: Similar to the last problem, we get an underdamped system which is only consistent with GRAPH (C).

(f) $y'' + 3y = \cos(11t)$

SOLUTION: In this case, $\sqrt{3}$ is really not close to 11, so we don't expect this to produce the beats in graph (d). Compare with (h).

(g) $y'' + 9y = 0$

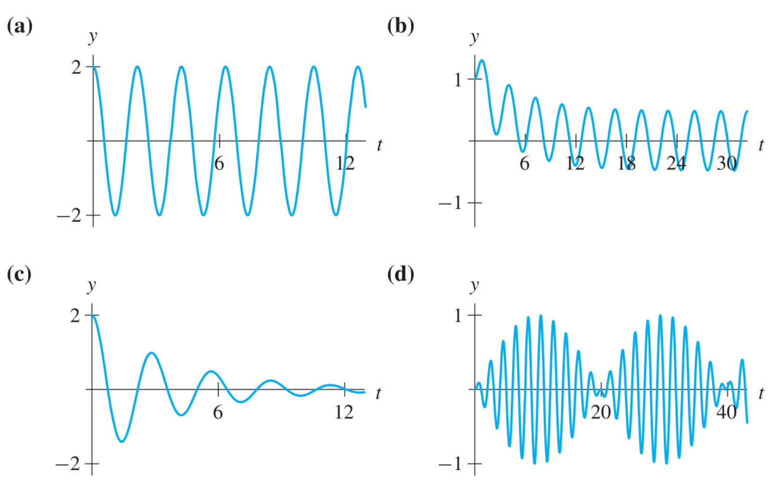
SOLUTION: Compare with (a) above, but this has period $2\pi/3$. This is consistent with GRAPH (A).

(h) $y'' + 11y = \cos(3t)$

SOLUTION: $\sqrt{11}$ is close to 3, so we do expect beats here. As an extra part of the analysis, we might also note that the period of the beat should be:

$$\frac{1}{2}\frac{2\pi}{(\omega - b)/2} = \frac{2\pi}{|\sqrt{11} - 3|} \approx \frac{2\pi}{0.32} \approx 19$$

which is not something I would expect you to compute without a calculator.



5. Rather than give you the second order linear differential equation, below are provided the solutions to the characteristic equation and the forcing function.

   Give the **form** of the solution to the particular part, but do NOT solve for the coefficients.

   (a) $\lambda = -1, -2$, $F(t) = t^2 + 3t$
   SOLUTION: $y_p = At^2 + Bt + C$

   (b) $\lambda = -2 \pm i$, $F(t) = \mathrm{e}^{-2t}$
   SOLUTION: $y_p = A\mathrm{e}^{-2t}$

   (c) $\lambda = -2 \pm i$, $F(t) = t^2\mathrm{e}^{-2t}$
   SOLUTION: $y_p = \mathrm{e}^{-2t}(At^2 + Bt + C)$

   (d) $\lambda = -2, -2$, $F(t) = t\mathrm{e}^{-2t}$
   SOLUTION: Initially, we would guess $(At + B)\mathrm{e}^{-2t}$, but this is the homogeneous equation. Multiplying by t leaves the term $Bt\mathrm{e}^{-2t}$ which is still part of the homogeneous equation, so multiply by t again:
   $$y_p = t^2(At + B)\mathrm{e}^{-2t}$$

   (e) $\lambda = -1 \pm 8i$, $F(t) = \mathrm{e}^{-t}\cos(3t)$
   SOLUTION: We can take $y_p = A\mathrm{e}^{(-1+3i)t}$, then take the real part of the solution. Alternatively,
   $$y_p = \mathrm{e}^{-t}(A\cos(3t) + B\sin(3t))$$

6. Given $y'' + 2y' + 10y = \mathrm{e}^{-2t}\sin(2t)$, is there a way to "complexify" this problem? Solve the DE using the complexification.

   SOLUTION: The homogeneous part of the solution is
   $$(\lambda^2 + 2\lambda + 1) + 9 = 0 \quad \Rightarrow \quad \lambda = -1 \pm 3i$$
   so that
   $$y_h = C_1\mathrm{e}^{-t}\cos(3t) + C_2\mathrm{e}^{-t}\sin(3t)$$
   For the particular solution, we would guess:
   $$y_p(t) = A\mathrm{e}^{(-2+2i)t} \qquad (\text{ Take the imag part})$$
   Put this into the DE:
   $$A\mathrm{e}^{(-2+2i)t}((-2+2i)^2 + 2(-2+2i) + 10) = \mathrm{e}^{(-2+2i)t}$$
   Expanding this out,
   $$A = \frac{1}{6-4i} = \frac{6+4i}{52} = \frac{3}{26} + \frac{1}{13}i$$
   Now, we want the imaginary part of $A\mathrm{e}^{(-2+2i)t}$. Below, I factor out e^{-2t}
   $$\text{imag}\left(\mathrm{e}^{-2t}\left(\frac{3}{26} + \frac{1}{13}i\right)(\cos(2t) + i\sin(2t))\right) = \mathrm{e}^{-2t}\left(\frac{3}{26}\sin(2t) + \frac{1}{13}\cos(2t)\right)$$

7. Given $y'' + 2y' + y = \cos(2t) - 2\sin(2t)$, is there a way we can complexify the problem?

   SOLUTION: Take $y'' + 2y' + y = \mathrm{e}^{3it}$, so we can solve for y_{p_1} when the forcing function is $\cos(3t)$ and y_{p_2} for when the forcing function is $\sin(3t)$.

   The overall forcing function would then be $y_{p_1} - 2y_{p_2}$ (this is from extended linearity).

   Here we go- Substuting $A\mathrm{e}^{3it}$ in for y, we end up with:
   $$A(-9 + 2(3i) + 1) = 1 \quad \Rightarrow \quad A = \frac{1}{-8+6i} = -\frac{8}{100} - \frac{6}{100}i = -\frac{2}{25} - \frac{3}{50}i$$

Therefore,

$$y_{p_1} = \text{Real}\left(\left(-\frac{2}{25} - \frac{3}{50}\,i\right)(\cos(3t) + i\sin(3t))\right) = -\frac{2}{25}\cos(3t) + \frac{3}{50}\sin(3t)$$

Therefore,

$$y_{p_2} = \text{Imag}\left(\left(-\frac{2}{25} - \frac{3}{50}\,i\right)(\cos(3t) + i\sin(3t))\right) = -\frac{3}{50}\cos(3t) - \frac{2}{25}\sin(3t)$$

Therefore, the overall particular part:

$$y_{p_1} - 2y_{p_2} = \frac{1}{25}\cos(3t) + \frac{11}{50}\sin(3t)$$

so that the general solution is:

$$\mathrm{e}^{-t}(C_1 + C_2 t) + \frac{1}{25}\cos(3t) + \frac{11}{50}\sin(3t)$$

8. For the following, just find the amplitude and phase angle of the steady state solution (you should not need to find the general solution to the DE). You may use a calculator on these- on the exam I will make an extra effort to find nice numbers.

   (a) $y'' + 6y' + 13y = 2\cos(3t)$
   SOLUTION: As is our usual practice, let $y_p = A\mathrm{e}^{3it}$. Putting it into the DE yields:

   $$A\mathrm{e}^{3it}(-9 + 6(3i) + 13) = 2\mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{2}{4 + 18i}$$

   The amplitude is then

   $$2\frac{1}{|4 + 18i|} = \frac{2}{\sqrt{4^2 + 18^2}} = \frac{2}{\sqrt{340}} = \frac{1}{\sqrt{85}}$$

   (I'll try to be sure the numbers on the exam work out OK). The phase angle is

   $$\delta = \tan^{-1}\left(\frac{9}{2}\right)$$

   We would not need to add π.

   (b) $y'' + 2y' + 3y = \cos(2t)$
   SOLUTION: Substitute $y_p = A\mathrm{e}^{2it}$, so the DE becomes:

   $$A\mathrm{e}^{2it}(-4 + 2(2i) + 3) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-1 + 4i}$$

   The amplitude of y_p is then $1/\sqrt{5}$, and the phase angle will be

   $$\delta = \tan^{-1}\left(-\frac{4}{1}\right) + \pi$$

   We add π because the point $-1 + 4i$ is in quadrant II (the inverse tangent would return an angle in quadrant IV).

   (c) $y'' + 4y' + 4y = 2\cos(3t)$
   SOLUTION: Substitute $y_p = A\mathrm{e}^{3it}$ so that the DE becomes:

   $$A\mathrm{e}^{3it}(-9 + 4(3i) + 4) = 2\mathrm{e}^{3it}$$

   so that

   $$A = \frac{2}{-5 + 12i} \quad \Rightarrow \quad |A| = \frac{2}{\sqrt{25 + 144}} = \frac{2}{\sqrt{169}} = \frac{2}{13}$$

   with phase angle, as before,

   $$\delta = \tan^{-1}\left(-\frac{12}{5}\right) + \pi$$

   where again, we add π because the point $-5 + 12i$ is in quadrant II.

9. Consider: $y'' + y' + 2y = \cos(\omega t)$.

   - Write the amplitude of the forced response in terms of ω.
     SOLUTION: As in the previous problem, take $y_p = A\mathrm{e}^{i\omega t}$ so that

     $$A\mathrm{e}^{i\omega t}(-\omega^2 + i\omega + 2) = \mathrm{e}^{i\omega t} \quad \Rightarrow \quad A = \frac{1}{(2-\omega^2) + i\omega} \quad \Rightarrow \quad |A| = \frac{1}{\sqrt{(2-\omega^2)^2 + \omega^2}}$$

   - Find ω that maximizes the amplitude.
     Take the derivative of $|A|$ with respect to ω. Recall that in class, we showed that this meant we could just take the derivative of the quantity under the radical sign:

     $$\frac{d}{d\omega}((2-\omega^2)^2 + \omega^2) = 2(2-\omega^2)^1(-2\omega) + 2\omega = 0$$

     We note that $\omega \neq 0$, so we can divide the common 2ω out:

     $$-2(2-\omega^2) + 1 = 0 \quad \Rightarrow \quad -4 + 2\omega^2 + 1 = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{3}{2}}$$

     (We assume that ω is positive- The cosine is an even function anyway).

10. Go back to the differential equations in problem 2. Label each as either overdamped, underdamped, critically damped, or N/A.

    SOLUTION: In order, these would be critically damped, overdamped, N/A, underdamped.

11. Write $-\cos(3t) + \sqrt{3}\sin(3t)$ as $R\cos(\omega t - \delta)$.
    SOLUTION:

    $$R = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2 \qquad \delta = \tan^{-1}\left(-\sqrt{3}\right) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

12. Use complexification to find trig identities for $\cos(x-y)$ and $\sin(x-y)$.
    SOLUTION: We can get both, since

    $$\cos(x-y) + i\sin(x-y) = \mathrm{e}^{i(x-y)} = \mathrm{e}^{ix}\mathrm{e}^{-iy} = (\cos(x) + i\sin(x))(\cos(y) - i\sin(y)) =$$

    $$(\cos(x)\cos(y) + \sin(x)\sin(y)) + i(\sin(x)\cos(y) - \sin(y)\cos(x))$$

    so that

    $$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

    and

    $$\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x)$$

13. Below are four graphs and six differential equations. Match each graph to its appropriate differential equation (this is Exercise 21, Section 4.3).

    SOLUTION: Before we start, graph (a) represents resonance, which is represented only in Equation (e), and we see beating in (b), (d). The last one is a pure periodic function with a positive constant forcing, so that's Equation (a).

    That leaves only equations (c) and (d) to compare with graphs (b) and (d). This is where it is nice to know the period of the beats, which is

    $$\frac{1}{2}\frac{2\pi}{|\omega - b|/2}$$

    In Equation (c), the period would be 2π, so that's graph (b), and that leaves graph (d) to be paired with Equation (d).

(a) $y'' + 16y = 10$

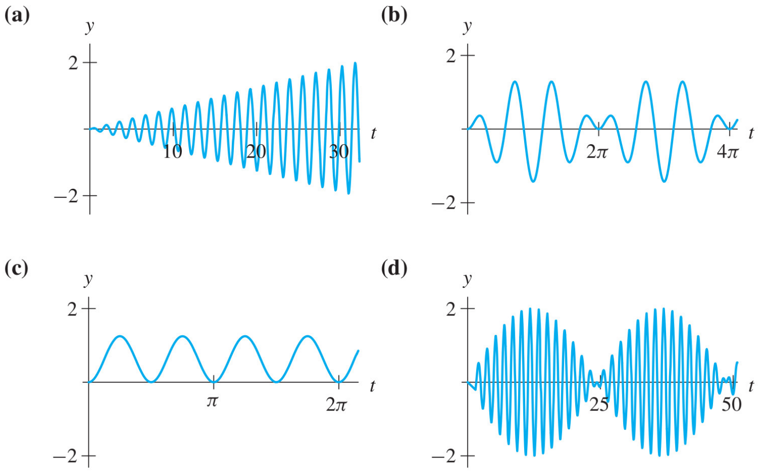
(b) $y'' + 16y = -10$

(c) $y'' + 16y = 5\cos(3t)$

(d) $y'' + 14y = 2\cos(4t)$

(e) $y'' + 16y = \frac{1}{2}\cos(4t)$

(f) $y'' + 2y' + 16y = \cos(4t)$



14. Suppose that two species, x and y are to be introduced to an island. It is known that the two species compete, but the precise nature of this interaction is unknown. We assume that $x(t)$ and $y(t)$ are modeled by a system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

If it is known that each species reproduces very slowly and there is intense competition between them, answer the following questions.

TYPO: I should have added that $(0,0)$ is a known equilibrium.

Before we answer, notice that the linearized system of equations is given by:

$$\begin{aligned} x' &= f_x(0,0)x + f_y(0,0)y \\ y' &= g_x(0,0)x + g_y(0,0)y \end{aligned}$$

(a) What can we conclude about f_x and g_y at $(0,0)$?
SOLUTION: Since $x' \approx f_x(0,0)x$ and $y' \approx g_y(0,0)y$, then these values will be small and positive.

(b) What can we conclude about f_y and g_x at $(0,0)$?
SOLUTION: Since an increase in y would cause dx/dt to fall, we expect f_y to be negative and large in magnitude. Similarly, g_x would follow as well.

(c) Using the assumptions from the previous two answers, see if you can classify the origin using the Poincaré Diagram.
SOLUTION: Consider the matrix with two small positive numbers along in the position of f_x, g_y and large negative numbers in f_y, g_x. For example:

$$\begin{bmatrix} 1/10 & -10 \\ -20 & 1/5 \end{bmatrix}$$

The trace will be a small positive number, the determinant will be negative, so the origin will be a saddle. This would actually be an odd situation, but there you are.

15. Short Answer:

(a) What is the extended linearity principle?

SOLUTION: Given $ay''+by'+cy = F_1(t)+F_2(t)+\cdots+F_k(t)$, we solve it by adding in the solutions to the homogeneous equation (y_h), then add in the particular solutions, where y_{p_i} solves the DE with forcing function $F_i(t)$, so that the overall solution is given by:

$$y_h(y) + y_{p_1}(t) + \cdots y_{p_k}(t)$$

(b) What is "beating"? How do we find the period of a beat?

Beating occurs in the undamped, periodically forced oscillator when the period of the forcing function gets close to the natural period. The period of a beat is:

$$\frac{1}{2}\frac{2\pi}{|\omega - b|/2} = \frac{2\pi}{|\omega - b|}$$

(c) What is "resonance"? (In both undamped and damped systems).

Resonance in the undamped, periodically forced oscillator will occur when the frequency (or period) of the forcing function is equal to the frequency of the homogeneous solution.

In the damped oscillator, "resonance" will still occur, but as a response that has been tuned to maximize its amplitude with respect to its frequency- Like breaking a glass with your voice.

(d) Find all equilibrium solutions to $y'' + 4y = \sin(t)$.

SOLUTION: This is a bit of a "trick question". Equilibrium solutions are solutions that never change in time (so that would make them constant). There are no constant solutions to this DE.

(e) What is the frequency of the steady state solution to the equation:

$$y'' + 3y' + y = 4\cos(2t)$$

SOLUTION: The frequency will be the same as the forcing function, $2/(2\pi) = 1/\pi$.

(f) Consider the following 3 equations, each of which has an equilibrium solution at the origin. Which two systems have phase portraits with the same local picture? Justify your answer.

(i) $\begin{aligned} x' &= 3\sin(x) + y \\ y' &= 4x + \cos(y) - 1 \end{aligned}$ (ii) $\begin{aligned} x' &= -3\sin(x) + y \\ y' &= 4x + \cos(y) - 1 \end{aligned}$ (iii) $\begin{aligned} x' &= -3\sin(x) + y \\ y' &= 4x + 3\cos(y) - 3 \end{aligned}$

SOLUTION: Consider the linearizations for each- The matrix for the last two will be identical, and different from the first.

16. For each system, (a) find and classify the equilibria, (b) sketch the nullclines, and (c) give a general sketch of the phase portrait.

(a) $\begin{aligned} x' &= x - 3y^2 \\ y' &= x - 3y - 6 \end{aligned}$

SOLUTION: We can look at the nullclines first, or solve for the equilibria first. If we get the equilibria first, we solve:

$$\begin{aligned} x - 3y^2 &= 0 \\ x - 3y - 6 &= 0 \end{aligned} \quad \Rightarrow \quad x = 3y^2 \quad \Rightarrow \quad 3y^2 - 3y - 6 = 0 \quad \Rightarrow \quad y = -1, 2$$

For $y = -1$, $x = 3$ (for the point $(3, -1)$) and for $y = 2$, $x = 12$ (for the point $(12, 2)$). The plot of the first nullcline is a sideways parabola, the second a line (see below).

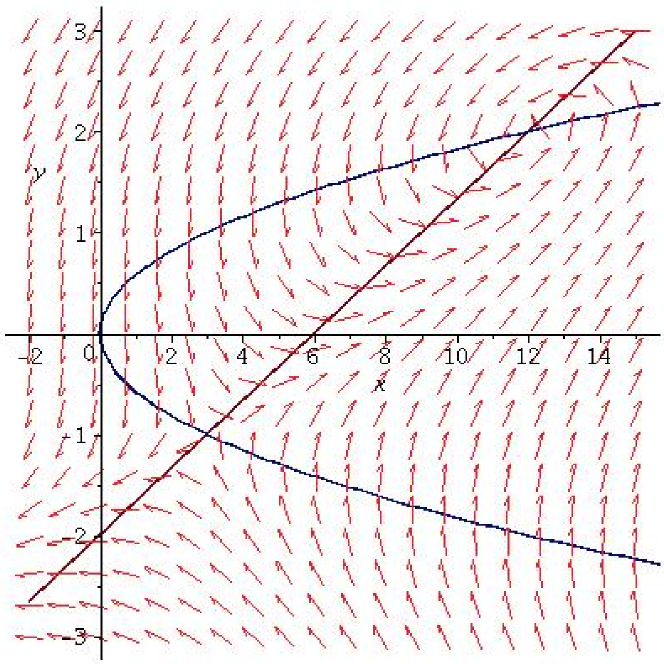
Before we do the nullcline analysis, we can classify the equilibria. Computing the Jacobian, we have:

$$J = \begin{bmatrix} 1 & -6y \\ 1 & -3 \end{bmatrix} \quad \Rightarrow \quad J(3,-1) = \begin{bmatrix} 1 & 6 \\ 1 & -3 \end{bmatrix}, \quad J(12,2) = \begin{bmatrix} 1 & -12 \\ 1 & -3 \end{bmatrix}$$

At $(3, -1)$ you should find it is a SADDLE, and $(12, 2)$ is a SPIRAL SINK. Continuing the analysis using nullclines-

- If we're along the line, inside the parabola, then $y' = 0$, and for x', find a sample point inside the parabola, like $(1, 0)$. Then $1 - 0 > 0$, so $x' > 0$.
- If we're along the line and outside the parabola, then $y' = 0$ and $x' < 0$ (opposite the last item).
- If we're along the parabola, and under the line, $x' = 0$, and for y', choose a sample point like $(0, -3)$ which gives $y' > 0$.
- If we're along the parabola and over the line, then $x' = 0$ and $y' < 0$.

Below is the graph of the nullclines together with the direction field.



(b) $$\begin{aligned} x' &= 10 - x^2 - y^2 \\ y' &= 3x - y \end{aligned}$$

This one works in much the same way as the previous one. In this case, we have a circle and a line. The equilibria are:

$$10 - x^2 - (3x)^2 = 0 \quad \Rightarrow \quad x = \pm 1, y = \pm 3$$

To be more precise, we have two equilibria, $(1, 3)$ and $(-1, -3)$. The Jacobian matrix is below:

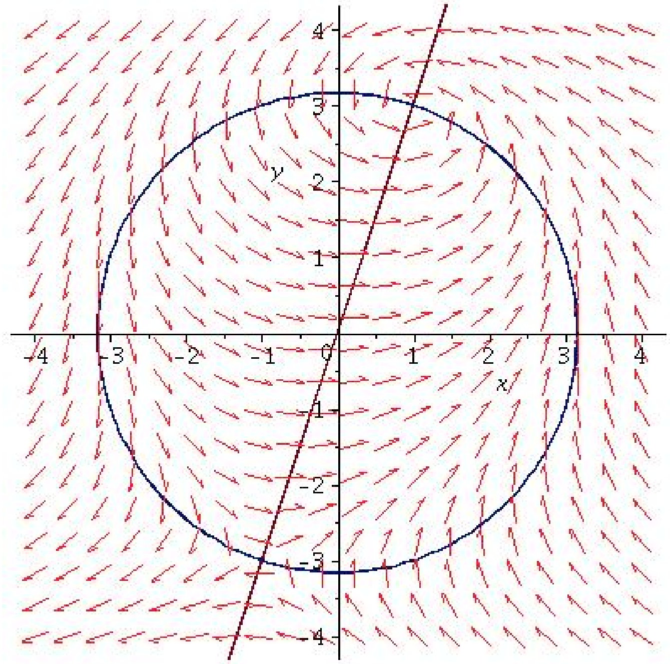
$$J = \begin{bmatrix} -2x & -2y \\ 3 & -1 \end{bmatrix} \quad \Rightarrow \quad J(1,3) = \begin{bmatrix} -2 & -6 \\ 3 & -1 \end{bmatrix}, \quad J(-1,-3) = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

We should find that at $(1, 3)$ we have a spiral sink, and at $(-1, -3)$ we have a saddle.

Doing the nullcline analysis:

- If we're on the circle, then $x' = 0$, and we're then either above the line or below the line. Use sample points to check (for example, $(0, \pm 1)$). We see that points above the line mean $y' < 0$ and points below the line mean $y' > 0$.
- If we're on the line, then we're inside or outside the circle. Inside the circle, $x' > 0$ and outside the circle $x' < 0$.

Putting all this together, we should see something like the figure below, which is a plot of the nullclines together with the full direction field.



17. Consider the differential equation:

$$x'' + 2x' - 3x + x^3 = 0$$

(a) Convert this to a system of first order equations.
SOLUTION: Let $x_1 = x$ and $x_2 = x'$. You can also translate that into x, y which may be easier for you to compute partial derivatives.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1^3 + 3x_1 - 2x_2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x' &= y \\ y' &= -x^3 + 3x - 2y \end{aligned}$$

For equilibria, we get $(0,0)$, $(\sqrt{3},0)$ and $(-\sqrt{3},0)$.

(b) Classify those equilibria using linearization.
SOLUTION: The Jacobian is given by:

$$J = \begin{bmatrix} 0 & 1 \\ -3x^2+3 & -2 \end{bmatrix}, \quad J(0,0) = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}, \quad J(\pm\sqrt{3},0) = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix}$$

The origin is a saddle, both the other equilibria are spiral sinks. Just for fun, below is the null-cline/direction field plot.