The Study of Systems of Differential Equations

In Chapter 1 we learned that there are three basic ways to understand the solutions of a differential equation—with the use of analytic, geometric (or qualitative), and numeric techniques. In the subsequent sections of this chapter, we will concentrate on analogous approaches for systems and second-order equations. In the next section we introduce vector notation in order to provide a geometric approach. In Sections 2.3 and 2.4, we discuss analytic techniques that we can use to find explicit formulas for solutions in somewhat specialized situations, and in Section 2.5 we generalize Euler's method to systems of differential equations.

EXERCISES FOR SECTION 2.1

Exercises 1–6 refer to the following systems of equations:

(i)
$$\frac{dx}{dt} = 10x\left(1 - \frac{x}{10}\right) - 20xy$$
$$\frac{dy}{dt} = -5y + \frac{xy}{20}$$

(ii)
$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$
$$\frac{dy}{dt} = 15y\left(1 - \frac{y}{15}\right) + 25xy.$$

1. In one of these systems, the prey are very large animals and the predators are very small animals, such as elephants and mosquitoes. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey. Determine which system is which and provide a justification for your answer.

2. Find all equilibrium points for the two systems. Explain the significance of these points in terms of the predator and prey populations.

3. Suppose that the predators are extinct at time $t_0 = 0$. For each system, verify that the predators remain extinct for all time.

4. For each system, describe the behavior of the prey population if the predators are extinct. (Sketch the phase line for the prey population assuming that the predators are extinct, and sketch the graphs of the prey population as a function of time for several solutions. Then interpret these graphs for the prey population.)

5. For each system, suppose that the prey are extinct at time $t_0 = 0$. Verify that the prey remain extinct for all time.

6. For each system, describe the behavior of the predator population if the prey are extinct. (Sketch the phase line for the predator population assuming that the prey are extinct, and sketch the graphs of the predator population as a function of time for several solutions. Then interpret these graphs for the predator population.)