Lecture Notes: Complex Eigenvalues

Today we consider the second case when solving a system of differential equations by looking at the case of complex eigenvalues.

Last time, we saw that, to compute eigenvalues and eigenvectors for a matrix A, we first compute the characteristic equation, then solve for a representative eigenvector.

We applied this to $\mathbf{x}' = A\mathbf{x}$ for "Case 1" which was when we had two distinct real eigenvalues, $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$, and saw that the general solution is:

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

For today, let's start by looking at the eigenvalue/eigenvector computations themselves in an example. For the matrix A below, compute the eigenvalues and eigenvectors:

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

SOLUTION: You don't necessarily need to write the first system to the left, but definitely write the one to the right:

$$\begin{array}{rl} 3v_1 - 2v_2 & = \lambda v_1 \\ v_1 + v_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rl} (3-\lambda)v_1 - 2v_2 & = 0 \\ v_1 + (1-\lambda)v_2 & = 0 \end{array}$$

Form the characteristic equation using the shortcut or by taking the determinant of the coefficient matrix. Please complete the square when solving for λ!

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = 2 \pm i$$

Now, if $\lambda = 2 + i$, solve for an eigenvector:

$$\begin{array}{rrl} (3-(2+i))v_1 & -2v_2 & = 0 \\ v_1 & +(1-(2+i))v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (1-i)v_1 & -2v_2 & = 0 \\ v_1 & +(-1-i)v_2 & = 0 \end{array}$$

Side Note/Side Computation

Recall that we said that these equations needed to be the same line- Indeed they are. To see this, if you divide the first equation by $1 - i$, we get:

$$\frac{1-i}{1-i}v_1 - \frac{2}{1-i}v_2 = 0 \quad \Rightarrow \quad v_1 - \frac{2(1+i)}{(1^2+1^2)}v_2 = 0 \quad \Rightarrow v_1 - (1+i)v_2 = 0$$

which is the second equation.

Returning to the Problem...

Remembering two shortcuts: (i) we only need to compute the first equation, and (ii) if we have $av_1 + bv_2 = 0$, take the eigenvector to be $(b, -a)$ or $(-b, a)$, then let's finish this problem up:

Given $(1-i)v_1 - 2v_2 = 0$, we can use $\mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$.

As a side remark, the other eigenvalue/eigenvector pair are the complex conjugates (we won't be using them):

$$\lambda_2 = 2 - i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

The next section tells us how to solve the system.

Applying Complex evals to Systems of DEs

Suppose we have a complex eigenvalue, $\lambda = a \pm ib$. Use one of them to construct the corresponding eigenvector (complex) $\mathbf{v}$. We can then solve the system using the theorem below.

Theorem: Given $\lambda = a + ib$, $\mathbf{v}$ for a matrix A in $\mathbf{x}' = A\mathbf{x}$, the solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

Notice that this is the extension of what we did in Chapter 3.

Example

Give the general solution to the system $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{x}$

This is the system for which we already have the eigenvalues and eigenvectors:

$$\lambda = 2 + i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Now, compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(2+i)t} \begin{bmatrix} 2 \\ 1-i \end{bmatrix} = \mathrm{e}^{2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 2 \\ 1-i \end{bmatrix} =$$

$$\mathrm{e}^{2t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ (\cos(t) + \sin(t)) + i(-\cos(t) + \sin(t)) \end{bmatrix}$$

so that the general solution is given by:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix} + C_1 \mathrm{e}^{2t} \begin{bmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{bmatrix}$$

From Chapter 3...

As a side remark, if I had solved the second equation for x_1 and substituted it into the first, I would have had:

$$x_2'' - 4x_2' + 5x_2 = 0 \quad \Rightarrow \quad r = 2 \pm i \quad \Rightarrow \quad x_2 = C_1 e^{2t}\cos(t) + C_2 e^{2t}\sin(t)$$

Example

Give the general solution to the system: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

First, the characteristic equation: $\lambda^2 + 1 = 0$, so that $\lambda = \pm i$.

Now we solve for the eigenvector to $\lambda = i$:

$$\begin{aligned} (2-i)v_1 - 5v_2 &= 0 \\ 1v_1 + (-2-i)v_2 &= 0 \end{aligned}$$

Using the second equation, $v_1 - (2+i)v_2 = 0$, and we have our eigenvalue/eigenvector pair. Now we compute the needed quantity, $e^{\lambda t}\mathbf{v}$:

$$e^{it}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos(t) + i\sin(t))(2+i) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

Simplifying, we get:

$$\begin{bmatrix} (2\cos(t) - \sin(t)) + i(2\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

The solution is:

$$\mathbf{x}(t) = C_1\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2\begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

From Chapter 3...

We will quickly verify that this is what we would get using the techniques of Chapter 3. From the second equation, solve for x_1, then use the first equation to get a second order DE for x_2.

$$x_1 = x_2' + 2x_2 \quad \Rightarrow \quad (x_2'' + 2x_2') = 2(x_2' + 2x_2) - 5x_2 \quad \Rightarrow \quad x_2'' + x_2 = 0$$

Therefore, $x_2 = C_1\cos(t) + C_2\sin(t)$. Solving for x_1:

$$x_1 = x_2' + 2x_2 = (-C_1\sin(t) + C_2\cos(t)) + 2(C_1\cos(t) + C_2\sin(t))$$

and we see that we get the identical solution.

Graphically, the solutions are ellipses. In fact, if we solve the differential equation by computing dy/dx, we get solutions of the form:

$$x^2 - 4xy + 5y^2 = C$$

Graphical Analysis of Solutions

We might take note of the magnitude of our solution,

$$|\mathbf{x}(t)| = |e^{\lambda t}||\mathbf{v}|$$

The magnitude of the eigenvector is some constant, so the magnitude of the solution depends on the magnitude of $e^{\lambda t}$. If we assume that $\lambda = \alpha + \beta i$, then

$$|e^{\lambda t}| = |e^{\alpha t}||\cos(\beta t) + i\sin(\beta t)|$$

Since $(\cos(\beta), \sin(beta))$ is a point on the unit circle, this says that the magnitude of our solution completely depends on $e^{\alpha t}$, or more precisely, α, which is the real part of λ.

- If $\alpha < 0$, solutions tend towards the origin as $t \to \infty$.
- If $\alpha = 0$, solutions are bounded (the solution is a closed curve, either a circle or an ellipse).
- If $\alpha > 0$, solutions tend to $\pm\infty$ as $t \to \infty$.

Also, in every case, $\beta \neq 0$ (otherwise λ would not be complex). This will introduce a rotation into the solutions. Putting this together with the three notes above, we get the summary below:

Graphical Summary- Complex Eigenvalues

Notice that if the real part of λ is positive, solutions "blow up". If the real part of λ is negative, $y(t) \to 0$ as $t \to \infty$. Therefore, the origin can be classified by $\lambda = \alpha \pm \beta i$:

- If $\alpha = 0$, we get pure periodic solutions (the period depends on β).
- If $\alpha < 0$, the origin is a *spiral sink.*
- If $\alpha > 0$, the origin is a *spiral source.*

So what does β control? Since β is involved with the period of the periodic part of the solution, if we fix α and increase β, that will result in a "tighter" spiral. However, the full relationship between α, β and how tight the spiral is turns out to be relative. See the graphs below for a sense of how that works.

If I ask you to graph a spiral, you do NOT need to be completely accurate with how tight the spiral is- Mainly, just be sure your rotation is correct, and that you are moving in the right direction (sink versus source).

Speaking of which, we cannot determine the direction of the spiral (clockwise or counterclockwise) just be looking that λ- We will need the actual matrix for that. Here are two examples:

$$\begin{bmatrix} 1 & 20 \\ -20 & 1 \end{bmatrix} \text{ with } \lambda = 1 \pm 20i \qquad \text{versus} \qquad \begin{bmatrix} 1 & -20 \\ 20 & 1 \end{bmatrix} \text{ with } \lambda = 1 \pm 20i$$

To tell which direction we're traveling, try out a sample point. For example, $(1, 0)$ along the $x-$axis. From the direction field, we have:

$$\begin{bmatrix} 1 & 20 \\ -20 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -20 \end{bmatrix}$$

Therefore, from $(1, 0)$ we're moving one unit forward, 20 units down- Corresponding to clockwise motion. On the other hand,

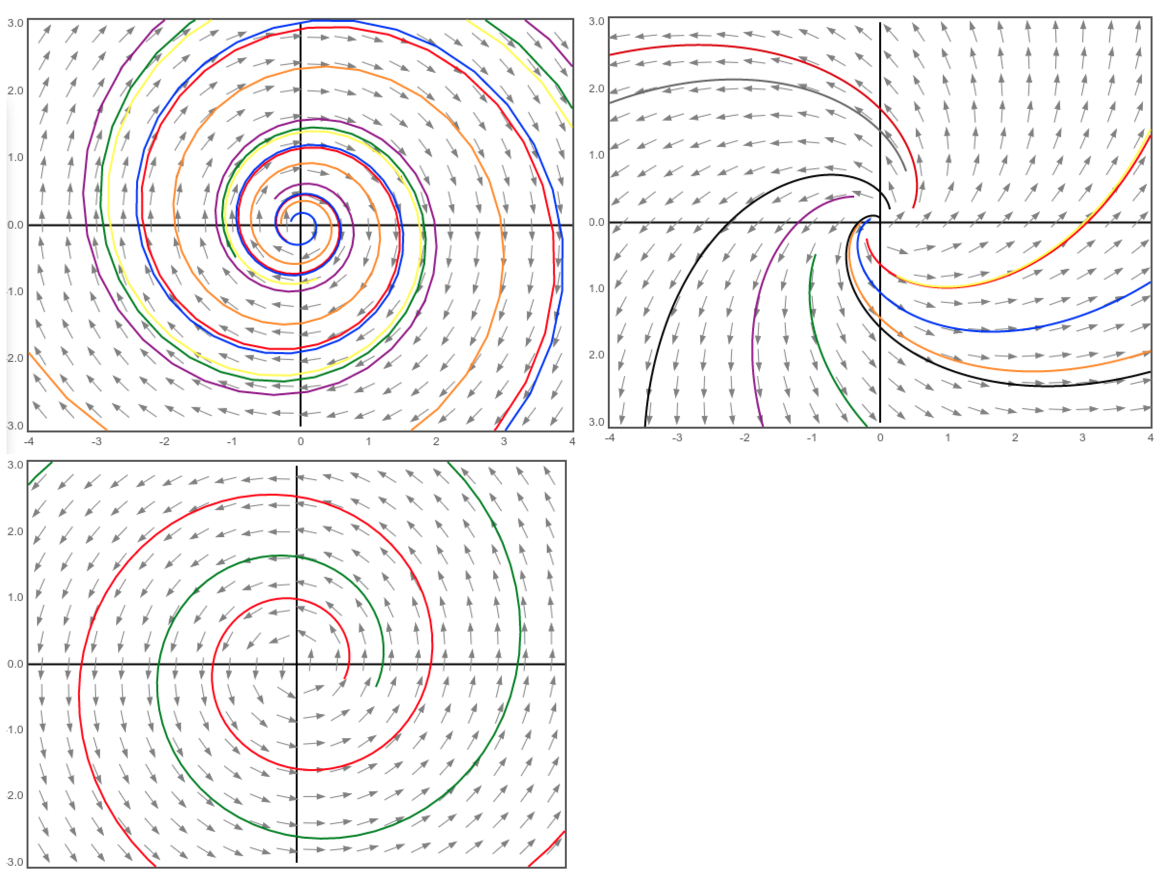
$$\begin{bmatrix} 1 & -20 \\ 20 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$$

And this means that, from the point $(1, 0)$, we travel in the direction of 1 unit forward, 20 units up. This corresponds to counterclockwise motion.

Summary: We cannot determine the direction of travel (CW or CCW) from λ. We have to go to the matrix.

We'll show some graphical solutions below so you get an idea. The three graphs below correspond to:

$$\lambda = 1 \pm 20i \qquad \lambda = 1 \pm i \qquad \lambda = \frac{1}{10} \pm i$$



One matrix with $\lambda = \alpha \pm \beta i$ is given by:

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

One of these has CW spin, one has CCW spin- Can you tell which is which?