# The Laplace Transform

$$\mathcal{L}(f(t)) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt = F(s)$$

## Example

$$\mathcal{L}(\mathrm{e}^{at}) = \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\, dt = \int_0^\infty \mathrm{e}^{-(s-a)t}\, dt = \left(-\frac{1}{s-1}\mathrm{e}^{-(s-a)t}\right|_0^\infty =$$

$$\frac{1}{s-a}, s > a$$

Issues to explore:

- For what $f$ does $\mathcal{L}$ exist?
- Recall how to take the limit (l'Hospital's Rule).
- Recall how to integrate by parts using a table.

# Some Laplace Transforms

We showed that $\mathcal{L}(\mathrm{e}^{at}) = \frac{1}{s-a}$. What is $\mathcal{L}(1) = \frac{1}{s}$

$$\mathcal{L}(t) = \int_0^\infty t\mathrm{e}^{-st}\,dt \quad \Rightarrow \quad \begin{array}{ccc} + & t & \mathrm{e}^{-st} \\ - & 1 & (-1/s)\mathrm{e}^{-st} \\ + & 0 & (1/s^2)\mathrm{e}^{-st} \end{array} \quad \Rightarrow$$

$$\lim_{T\to\infty}\left(-\frac{t\mathrm{e}^{-st}}{s} - \frac{\mathrm{e}^{-st}}{s^2}\right|_0^T = \frac{1}{s^2}$$

# Example

Use the computation for $\mathcal{L}(\mathrm{e}^{at})$ to compute

$$\mathcal{L}(\cos(at)) \quad \text{and} \quad \mathcal{L}(\sin(at))$$

SOLUTION: Complexify $\cos(at) + i\sin(at) = \mathrm{e}^{iat}$and so

$$\mathcal{L}(\mathrm{e}^{iat}) = \frac{1}{s - ia} = \frac{s + ia}{s^2 + a^2}$$

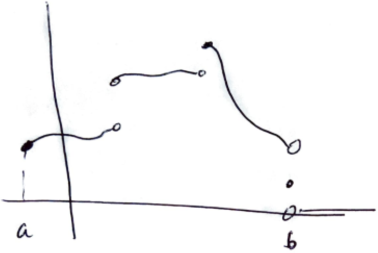
Therefore,

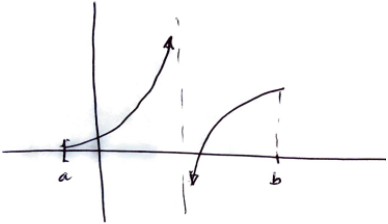
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2 + a^2} \qquad \mathcal{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

# A (Partial) Table of Transforms

| $f(t)$ | $F(s)$ | Notes |
|---|---|---|
| 1. $1$ | $\frac{1}{s} \quad s > 0$ | Sect 6.1 |
| 2. $e^{at}$ | $\frac{1}{s-a} \quad s > a$ | Sect 6.1 |
| 3. $t^n, \quad$ n pos int | $\frac{n!}{s^{n+1}}, \quad s > 0$ | Sect 6.1 |
| 4. $\sin(at)$ | $\frac{a}{s^2+a^2}, \quad s > 0$ | Sect 6.1 |
| 5. $\cos(at)$ | $\frac{s}{s^2+a^2}, \quad s > 0$ | Sect 6.1 |

A function is piecewise continuous, it is integrable.
Example on the left, not an example on the right:




PWC if finite set, $t_0, t_1, \ldots, t_n$ s.t.

- $a = t_0 < t_1 < t_2 < \cdots < t_n = b$
- $f$ is cont on each $t_i < t < t_{i+1}$.
- $\lim\limits_{t \to t_i^{+/-}} f(t)$ exists.

Def: $f$ is of exponential order if it does not grow faster than an exponential function. That is, we can find values of $M, a, t_0$ so that

$$|f(t)| \leq M\mathrm{e}^{at}, \qquad \text{for all } t > t_0$$

Examples:

- All bounded functions (like $\sin(t), \cos(t)$, etc)
  If $|f(t)| \leq M$ for all $t$, then $a = 0$ and $t_0$ is any number.
- All polynomials. Note that

  $$t^n = \mathrm{e}^{\ln(t^n)} = \mathrm{e}^{n\ln(t)} \leq \mathrm{e}^{nt} \qquad \text{for all } t > 0$$

  (Left as an exercise: $\ln(t) < t$ for all $t > 0$)
- "Not an example":

  $$\mathrm{e}^{t^2}$$

- Theorem 1:
  If $f$ is PWC and of exponential order, then $\mathcal{L}(f(t))$ exists.
- Theorem 2:
  $\mathcal{L}$ is a linear operator. That is, for any $f, g$ and constants $c$,

  $$\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t))+\mathcal{L}(g(t)) \qquad \mathcal{L}(c(f(t)) = c\mathcal{L}(f(t))$$

(Easy to prove- These two things are properties of the integral)

# Example

$$\mathcal{L}(3t - 2\mathrm{e}^{3t} + 5) = \mathcal{L}(3t) + \mathcal{L}(-2\mathrm{e}^{3t}) + \mathcal{L}(5) =$$

Continue, using linearity and the previous computations:

$$3\mathcal{L}(t) - 2\mathcal{L}(\mathrm{e}^{3t}) + 5\mathcal{L}(1) = 3 \cdot \frac{1}{s^2} - 2\frac{1}{s-3} + 5\frac{1}{s}$$

# The Inverse Transform

The inverse function to the Laplace transform exists as a complex integral, and so we will not formally define it. Rather, we will compute the inverse transform with the utilization of a few facts:

- The inverse Laplace transform exists.
- The inverse Laplace transform is (also) linear.
- A table of transforms will be provided- use it to invert.

Example: If $F(s) = \frac{s}{s^2+4}$, what is $f(t)$?

SOLUTION:
Looking at our list of transforms, we see that $f(t) = \cos(2t)$.
For homework examples from 6.1/6.2, go to the next video.