L004 Section 6.1 Examples, part 2 notes

In the last video, we looked at inverting transforms using partial fractions. Here we look at when and how to "complete the square" in the denominator.

- Find the inverse Laplace transform:

$$\frac{2s+3}{s^2+2s+5}$$

The denominator is irreducible (for example, $b^2-4ac = 2^2-4(5) < 0$). This is when we want to complete the square:

$$\frac{2s+3}{s^2+2s+5} = \frac{2s+3}{(s^2+2s+1)+4} = \frac{2s+3}{(s+1)^2+2^2}$$

We need to make this look like the table entries

$$\frac{s-a}{(s-a)^2+b^2} \qquad \frac{b}{(s-a)^2+b^2}$$

so we write:

$$\frac{2(s+1)+1}{(s+1)^2+2^2} = 2\frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

Now we can do the inversion:

$$2\mathrm{e}^{-t}\cos(2t) + \frac{1}{2}\mathrm{e}^{-t}\sin(2t)$$

- Suppose that we define

$$Y(s) = \int_0^\infty \mathrm{e}^{-st}y(t)\,dt = \mathcal{L}(y(t))$$

We want to write $\mathcal{L}(y'(t))$ in terms of $Y(s)$, if possible.

SOLUTION:

$$\mathcal{L}(y'(t))\,dt = \int_0^\infty \mathrm{e}^{-st}y'(t)\,dt \quad \Rightarrow \quad \begin{array}{c|c|c} + & \mathrm{e}^{-st} & y'(t) \\ - & -s\mathrm{e}^{-st} & y(t) \end{array} \quad \Rightarrow$$

$$\left(y(t)\mathrm{e}^{-st}\right|_0^\infty + s\int_0^\infty \mathrm{e}^{-st}y(t)\,dt$$

We assume that the limit of $y(t)\mathrm{e}^{-st}$ is zero (we assume $Y(s)$ exists), so that this expression simplifies to:

$$\mathcal{L}(y'(t)) = sY(s) - y(0)$$

- Similarly, find a formula for $\mathcal{L}(y''(t))$ in terms of $Y(s)$.

  SOLUTION: Almost identical to the previous problem, except the table has another row.

  SOLUTION:

$$\mathcal{L}(y''(t))\,dt = \int_0^\infty \mathrm{e}^{-st} y''(t)\,dt \quad \Rightarrow \quad \begin{array}{c|c|c} + & \mathrm{e}^{-st} & y''(t) \\ - & -s\mathrm{e}^{-st} & y'(t) \\ + & s^2\mathrm{e}^{-st} & y(t) \end{array} \quad \Rightarrow$$

$$\left(y'(t)\mathrm{e}^{-st} + s\mathrm{e}^{-st}y(t)\right|_0^\infty + s^2 \int_0^\infty \mathrm{e}^{-st} y(t)\,dt$$

  We assume that the limit of the expression is zero (we assume $Y(s)$ and $\mathcal{L}(y'(t))$ both exist), so that this expression simplifies to:

$$\mathcal{L}(y''(t)) = s^2 Y(s) - sy(0) - y'(0)$$

  These two expressions are special cases of Table Entry 18 (from the table in the book).