L006 Section 6.2 Examples

- Sectionb 6.2, #22: Solve

$$y'' - 2y' + 2y = \mathrm{e}^{-t} \qquad y(0) = 0 \quad y'(0) = 1$$

SOLUTION: Take the Laplace transform of both sides and simplify, solving for Y (then invert):

$$(s^2Y - s\cdot 0 - 1) - 2(sY - 0) + 2Y = \frac{1}{s+1} \quad \Rightarrow \quad (s^2 - 2s + 2)Y = \frac{1}{s+1} + 1$$

Note the appearance of the characteristic equation. Doing something slightly different than the video, I'll simplify the two expressions and perform the full partial fractions:

$$Y = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2} = \frac{s+2}{(s+1)(s^2-2s+2)}$$

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$

Continuing,

$$s - 2 = A(s^2 - 2s + 2) + (Bs + C)s = (A + B)s^2 + (-2A + C)s + 2A$$

Therefore,

$$\begin{array}{lrl} s^2: & 0 & = A + B \\ s: & 1 & = -2A + C \\ \mathit{const}: & -2 & = 2A \end{array} \quad \Rightarrow \quad A = -1, B = 1, C = -1$$

Now we have:

$$Y(s) = \frac{s+2}{(s+1)(s^2-2s+2)} = -\frac{1}{s+1} + \frac{s-1}{s^2-2s+2} = -\frac{s+1}{+}\frac{s-1}{(s-1)^2+1}$$

The solution to the IVP is given by the inverse Laplace transform:

$$y(t) = -\mathrm{e}^{-t} + \mathrm{e}^{t}\cos(t)$$