Introduction to Systems of DEs

Model: Rabbit ($x(t)$) and foxes ($y(t)$)

Assumption 1: In the absense of foxes, rabbit pop grows (exp growth)
In the absence of rabbits, fox pop declines (exp decline)
So far: $x' = \alpha x$ and $y' = -\gamma y$.

Assumption 2: The rate of change of the rabbit population is proportional to the number of rabbit-fox interactions.
Similarly, the fox population grows prop to number of rabbit-fox interactions.
In our model, xy will represent the total number of possible rabbit-fox interactions (one rabbit to one fox).

Model:

$$\begin{aligned} x' &= \alpha x - \beta xy \\ y' &= -\gamma y + \delta xy \end{aligned}$$

For example,

$$\begin{aligned} x' &= 2x - \tfrac{6}{5}xy \\ y' &= -y + \tfrac{9}{10}xy \end{aligned}$$

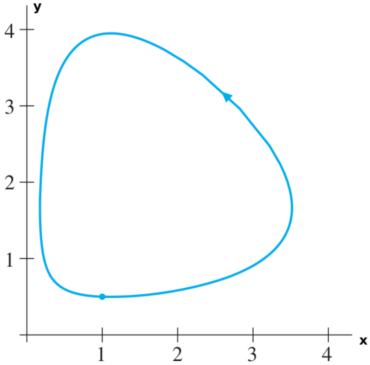
This is a nonlinear system of DEs.It is an autonomous system (no t).The solution is a parametric set of functions, $(x(t), y(t))$.

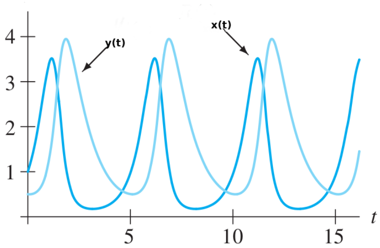
As before (Ch 2), given an autonomous DE, we can find *equilibrium solutions.* In this case, set the derivatives to zero and solve.

$$\begin{aligned} 0 &= x(2 - \tfrac{6}{5}y) \\ 0 &= y(-1 + \tfrac{9}{10}x) \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x = 0 \Rightarrow y = 0 &\text{ in Eqn 2} \\ y = \tfrac{5}{3} \Rightarrow x = \tfrac{10}{9} &\text{ in Eqn 2} \end{aligned}$$

Use the Java software from Canvas to see the plots (also in the YouTube video to follow).

Graphical Analysis





Also see the Java software on Canvas...

Exercise 1

$$\begin{aligned} x' &= 10x(1 - x/10) - 20xy \\ y' &= -5y + xy/20 \end{aligned} \qquad \begin{aligned} x' &= 0.3x - xy/100 \\ y' &= 15y(1 - y/15) + 25xy \end{aligned}$$

In one of these, prey is very large and predators are very small (like elephants and mosquitos). The other is very small prey and very large predators. Which is which?

Left: Small Prey, Right: Large prey.

Conversions

We can convert an n^{th} order DE to a system of first order. Example:

$$y'' + 3y' - 2y = 0$$

SOLUTION: Let $u = y$ and $v = y'$.Then the new system of DEs:

$$\begin{aligned} u' &= v \\ v' &= 2y - 3y' = 2u - 3v \end{aligned}$$

Conversions

Convert to a system of first degree equations:

$$y''' - 2yy' = \cos(t)$$

SOLUTION: Let $u = y$ and $v = y'$, and $w = y''$
Then the new system of DEs:

$$\begin{aligned} u' &= v \\ v' &= w \\ w' &= 2yy' + \cos(t) = 2uv + \cos(t) \end{aligned}$$

Example: Convert the second order DE to a system of first order:

$$y'' - 3y' - 2y = \mathrm{e}^{t}$$

SOLUTION:
Let $u = y, v = y'$. Then

$$\begin{aligned} u' &= v \\ v' &= 2y + 3y' + \mathrm{e}^{t} = 2u + 3v + \cos(t) \end{aligned}$$

Convert the following system to an equivalent 2d order DE:

$$\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= x_1 - x_2 \end{aligned}$$

SOLUTION: Take the first equation and solve for x_2 in terms of x_1.Put this substitution into the second equation to get a 2d order DE in terms of x_1.

$$x_2 = x_1' - 3x_1 \quad \Rightarrow \quad (x_1' - 3x_1)' = x_1 - (x_1' - 3x_1)$$

Simplify:

$$x_1'' - 3x_1' = x_1 - x_1' + 3x_1$$

$$x_1'' - 2x_1' - 4x_1 = 0$$

You try one!

Convert to a 2d order DE:

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}$$

(Press Pause!)

$$(x_1' + 2x_1)' = 3x_1 + 2(x_1' + 2x_1)$$

$$x_1'' - 7x_1 = 0$$

Conversions, Part 2

Given a first order system of DEs,

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

And, recalling from our Calculus III that, given parametric functions $(x(t), y(t))$, we can compute dy/dx:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Then, we might be able to solve the system by writing:

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

Example:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{y}$$

This is separable: $y\,dy = -x\,dx$, or

$$\frac{y^2}{2} = -\frac{x^2}{2} + C \quad \Rightarrow \quad x^2 + y^2 = C_2$$

Check: If we differentiate with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$