Linear Systems and Tanks (Replaces 7.1-7.2)

This is where we'll depart somewhat from the book. We will focus on systems of two equations in two unknowns to simplify our analysis.

Key Definition: A system of equations can be written in matrix-vector form as shown below (this is a definition)

$$\begin{array}{ll} ax + by & = e \\ cx + dy & = f \end{array} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Similarly, we could extend this to three variables (this is just to show you what it would look like):

$$\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & = b_3 \end{array} \quad \Leftrightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

More Definitions...

In Calc III, we defined a **vector** as something with direction and magnitude. For us, a vector is simply a column with a certain number of elements (the vectors above are in $\mathbb{R}^2$ since they each have two elements).

A **matrix** is simply an array of numbers, and the **size** of a matrix is defined as the number of *rows* $\times$ the number of *columns* (similar to a spreadsheet, rows always come first). We identify elements of the array by locating the row and column. For example, in the first matrix, if we call it A, then

$$\begin{array}{ll} A(1,1) = a & A(1,2) = b \\ A(2,1) = c & A(2,2) = d \end{array}$$

We will work with 2×2 matrices. The definition above gives meaning to **matrix-vector multiplication**. A couple of examples:

- Write the following system in equivalent matrix-vector form:

$$\begin{array}{rl} 3x - 2y & = 4 \\ x + y & = -1 \end{array} \qquad \text{Solution:} \qquad \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

- Using the definition, perform the matrix-vector multiplication:

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(2) \\ -1(1) + 3(2) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

From Calculus III, we already know how to compute the **determinant** of a 2×2 and a 3×3. There, we used straight lines as shortcut notation for the determinant (this is not the absolute value):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The **transpose** of a matrix A is denoted as A^T and is formed by taking the columns of A and making them the rows of A^T. The **trace** of a matrix is the sum of the diagonal elements. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad \mathrm{Tr}(A) = 1 + 4 = 5$$

Scalar Multiplication: Goes like you might suspect- Multiply every element of the matrix.

$$5 \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 5 & 10 \end{bmatrix}$$

Matrix-Matrix Multiplication is defined via matrix-vector multiplication. Think of the second matrix in terms of its columns:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} \right]$$

$$= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example: Compute the following:

$$\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -3 + 0 & -1 + 0 \\ 3 + 2 & 1 - 4 \end{bmatrix}$$

$$5 \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 5 & -10 \end{bmatrix}$$

Inverses and the Identity

There are two special matrices used in matrix multiplication: The identity and the inverse. The identity matrix is a matrix whose only non-zero elements are the ones along its diagonal. It can be any square size, as needed (use the one for which the given multiplication is defined).

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You will verify in the exercises that, for any matrix A, the identity works like the number 1 in the real numbers:

$$AI = IA = A$$

The inverse of a matrix A is another matrix, A^{-1} so that:

$$AA^{-1} = A^{-1}A = I$$

You will verify in the exercises that, given a 2×2 matrix, the inverse can be written down directly:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1}$$

Example: If $A \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, and λ is arbitrary scalar, compute $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$$

Solving the System

In solving a system of equations, there are three (and only three) possible outcomes: (i) Exactly one solution (intersecting lines), (ii) No Solution (parallel lines), (iii) an infinite number of solutions (the same line).

Theorem: If the matrix of coefficients has an inverse, then the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, $\mathbf{x} = A^{-1}\mathbf{b}$ (which could also be found by Cramer's Rule or computing the inverse directly using Equation 1).

Corollary 1: If the matrix of coefficients has a non-zero determinant, then there is exactly one solution to the system of equations (because we can compute the inverse).

Corollary 2: If we are solving $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$, then we obtain an infinite number of solutions *only* when $\det(A) = 0$ (You might notice that in this system, there are only two possible outcomes rather than three. What are they?)

Examples:

1. Solve the system:

$$\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

   SOLUTION: The determinant is -2, so there is exactly one solution. Below we solve it using the inverse (but you could use Cramer's Rule).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1/2 \end{bmatrix}$$

2. Solve the system:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

SOLUTION: The determinant is 0, so there is an infinite number of solutions (NOTE: We cannot have "no solution", because $x = 0$ and $y = 0$ is always one solution). The solutions are any (x, y) on either line (which is the same line):

$$\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

We **always** want to represent this in parametric form. To do this, we need a point (in this case, the origin is very nice), and a direction rather than a slope. Note that if the slope is m, the direction would be $\langle 1, m \rangle$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \quad \text{or} \quad t \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{or} \quad t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

There are an infinite number of ways of parameterizing the line- In the three cases above, the $t's$ are not equal to each other.

More on Lines and Rays

Recall from Calculus III: A line in two or three dimensions is defined by a **point** $\vec{p}$ and the **direction** $\vec{q}$:

$$\vec{p} + t\vec{q} \qquad -\infty < t < \infty$$

So, for example, the line going through the point $(1, 2, 3)$ in the direction of $\langle 1, -1, 1 \rangle$ can be written as:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + t \\ 2 - t \\ 3 + t \end{bmatrix}$$

Extra Example: What does this look like:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mathrm{e}^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad -\infty < t < \infty$$

SOLUTION: It is a ray rather than a line. As $t \to -\infty$, the length of the vector goes to zero (the line goes to the point), then as t increases, we move farther and farther in the direction given.

Systems of DEs and Matrices

Definition: An **autonomous** system of first order **linear** differential equations is a system of the following form. These are each equivalent to the other.

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

Definition: A **solution** to the system is a set of parametric functions that satisfies the given relationship.
Definition: The **trivial solution**: the origin $(x_1 = 0, x_2 = 0)$ is always a solution to the autonomous linear system. In fact, any constant solution to $A\mathbf{x} = \mathbf{0}$ is an **equilibrium solution.**

Examples

1. Show that $\mathbf{x}(t) = [\cos(t), \sin(t)]^T$ solves the system:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

SOLUTION: We compute $\mathbf{x}'(t)$ first, then we'll compute the matrix-vector on the right side of the equation. We want those two computations to be the same:

For the derivatives, we get $x_1'(t) = -\sin(t)$ and $x_2'(t) = \cos(t)$.

For the matrix-vector computation, we get:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

We see that they match.

2. Show that $\mathbf{x}(t) = \mathrm{e}^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ solves the differential equation:

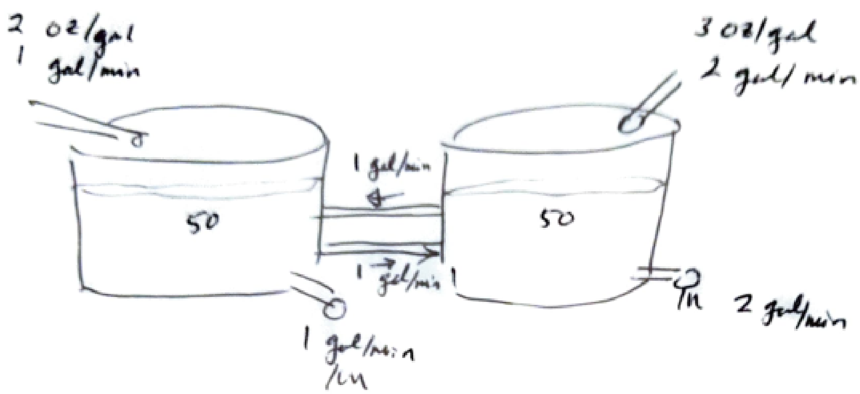
$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

SOLUTION: As before, first compute $\mathbf{x}'$, then compute $A\mathbf{x}$ and see if they are the same quantity:

- $\mathbf{x}' = 2\mathrm{e}^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \mathrm{e}^{2t} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$
- $A\mathbf{x} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathrm{e}^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \mathrm{e}^{2t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \mathrm{e}^{2t} \begin{bmatrix} 3(4) - 2(2) \\ 2(4) - 2(2) \end{bmatrix} = \mathrm{e}^{2t} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

Tank Mixing

Consider a system of two tanks, A and B. Initially, they both contain 50 gallons of pure water. A pipe flowing at 1 gal/min is pumping 2 oz/gal of salt into tank A, and is pumping brine at 2 gal/min with 3 oz/gal of salt into tank B. Further, there are tubes connecting tanks A and B, each is pumping at 1 gal/min. Lastly, a pipe leading out is pumping at 1 gal/min for tank A, and 2 gal/min from tank B (see the figure). Model the amount of salt in the tanks at time t.



SOLUTION: Remember to model (Rate of change) = Rate in − Rate out.

Let $A(t)$, $B(t)$ be the ounces of salt in Tanks A, B respectively. Then for tank A, we have the following. You might note that when brine is being pumped out, the destination doesn't really matter. For example, the "rate out" for tank A can be computed by combining the outputs to the outside and to tank B.

$$\frac{dA}{dt} = \left(\frac{2\text{ oz}}{\text{gal}} \cdot \frac{1\text{ gal}}{\text{min}} + \frac{1\text{ gal}}{\text{min}} \cdot \frac{B\text{ oz}}{50\text{ gal}}\right) - \left(\frac{2\text{ gal}}{\text{min}} \cdot \frac{A\text{ oz}}{50\text{ gal}}\right)$$

$$\frac{dB}{dt} = \left(\frac{3\text{ oz}}{\text{gal}} \cdot \frac{2\text{ gal}}{\text{min}} + \frac{1\text{ gal}}{\text{min}} \cdot \frac{A\text{ oz}}{50\text{ gal}}\right) - \left(\frac{3\text{ gal}}{\text{min}} \cdot \frac{B\text{ oz}}{50\text{ gal}}\right)$$

Simplifying a bit, we have:

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} -2/50 & 1/50 \\ 1/50 & -3/50 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

To find the equilibrium, we set the derivatives to zero. To simplify the equations, we'll also multiply by 50.

$$\begin{array}{rl} -2A + B + 100 & = 0 \\ A - 3B + 300 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} 2A - B & = 100 \\ -A + 3B & = 300 \end{array}$$

Using your favorite technique (substitution or Cramer's rule), we find that

$$A = 120 \qquad B = 140$$

You should check that these seem reasonable.

You might have noticed that we don't have the form $\mathbf{x}' = A\mathbf{x}$, but we're close. We can actually make our system look like this by making a small substitution:

$$\begin{aligned} x_1 &= A - 120 \\ x_2 &= B - 140 \end{aligned}$$

Now we create our system in x_1, x_2. First, we see that $x_1' = A'$ and $x_2' = B'$. Furthermore, we see that

$$\begin{bmatrix} -2/50 & 1/50 \\ 1/50 & -3/50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2/50 & 1/50 \\ 1/50 & -3/50 \end{bmatrix} \begin{bmatrix} A - 120 \\ B - 140 \end{bmatrix} = \begin{matrix} \frac{-2}{50}A + \frac{1}{50}B + 2 \\ \frac{1}{50}A - \frac{3}{50}B + 6 \end{matrix} = \begin{bmatrix} A' \\ B' \end{bmatrix}$$

Therefore, using the substitution $x_1 = A - 120$ and $x_2 = B - 140$, the equivalent system of equations is given by:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2/50 & 1/50 \\ 1/50 & -3/50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Homework (to replace 7.2)

1. Let A, B be the matrices below. Compute the matrix operation listed.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

   (a) $2A + B$ (c) BA (e) A^{-1}

   (b) AB (d) $A^T + B^T$ (f) B^{-1}

2. Vectors and matrices might have complex numbers. If $z = 3 + 2i$ and vector $\mathbf{v} = [1 + i, 2 - 2i]^T$, then find the real part and the imaginary part of $z\mathbf{v}$.

3. If a line goes through $(1, 2)$ in the direction of the vector $\langle -1, 1 \rangle$, write the equation of the line as $y = mx + b$.

4. Write the vector (parametric) form of the line (i) $y = 2x + 3$, (ii) $2x + 3y = 1$

5. Write the parametric form of the line through the point $(2, 3)$ with slope 2.

6. What will the graph of $e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be (where t is any real number).

7. Adding two vectors: Geometrically (and numerically) compute the following, where $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Be sure to draw each vector out, and see if you can see a pattern.

(a) $\mathbf{u}+\mathbf{v}$ (b) $\mathbf{u}-2\mathbf{v}$ (c) $\mathbf{u}+\frac{1}{2}\mathbf{v}$ (d) $-\mathbf{u}+\mathbf{v}$

8. Verify that $\mathbf{x}_1(t)$ below satisfies the DE below.

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}_1(t) = \mathrm{e}^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

9. Consider

$$\begin{aligned} x' &= 2x + 3y + 1 \\ y' &= x - y - 2 \end{aligned} .$$

First find the equilibrium solution, x_e, y_e. Then show that, if $u = x - x_e$ and $v = y - y_e$, then

$$\begin{aligned} u' &= 2u + 3v \\ v' &= u - v \end{aligned}$$

10. Each system below is *nonlinear*. Solve each by first writing the system as dy/dx.

(a) $\begin{aligned} x' &= y(1+x^3) \\ y' &= x^2 \end{aligned}$ (b) $\begin{aligned} x' &= 4 + y^3 \\ y' &= 4x - x^3 \end{aligned}$ (c) $\begin{aligned} x' &= 2x^2y + 2x \\ y' &= -(2xy^2 + 2y) \end{aligned}$

(Note: Some of these may be **exact**.)