Lecture Notes to substitute for 7.3-7.5

We want to solve the system:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \qquad \Rightarrow \qquad \mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

As we did in Chapter 3 for 2d order equations, we use an ansatz. In this case, the ansatz is $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, for some constant λ and some constant (non-zero) vector $\mathbf{v}$. Now, let's substitute the ansatz into our system of DEs and see what we get.

- First, the derivative with respect to t is straightforward, since $\mathbf{v}$ is a constant, and the exponential function is the only thing dependent on t. Therefore,

$$\mathbf{x}' = \lambda \mathrm{e}^{\lambda t}\mathbf{v}$$

- On the other side of the equation we have the matrix times $\mathbf{x}$, which gives us:

$$A\mathbf{x} = A\mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^{\lambda t}A\mathbf{v}$$

- Put the two sides together for $\mathbf{x}' = A\mathbf{x}$ and expand/simplify:

$$\mathrm{e}^{\lambda t}A\mathbf{v} = \lambda\mathrm{e}^{\lambda t}\mathbf{v} \quad \Rightarrow A\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \begin{aligned} av_1 + bv_2 = \lambda v_1 \\ cv_2 + dv_2 = \lambda v_2 \end{aligned}$$

This last system of equations, with λ and v_1, v_2 the unknowns, is a very important one- In fact, it represents an important **definition:**

If the system above is true for that particular value of λ and **non-zero** vector $\mathbf{v}$, then λ is an **eigenvalue** of the matrix A and $\mathbf{v}$ is an associated **eigenvector**. Note that while $\mathbf{v}$ is not allowed to be the zero vector, λ could be zero.

Continuing, Compute Eigenvalues and Eigenvectors:

Continuing with the system of equations, let's see how we actually solve for λ, v_1, v_2. First, we'll bring over the λ terms:

$$\begin{aligned} av_1 + bv_2 &= \lambda v_1 \\ cv_1 + dv_2 &= \lambda v_2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} (a-\lambda)v_1 \quad &+bv_2 &= 0 \\ cv_1 \quad &+(d-\lambda)v_2 &= 0 \end{aligned}$$

For this system to have a solution besides the trivial one ($v_1 = v_2 = 0$), we said earlier that the **determinant must be zero** (for example, so Cramer's rule cannot be used).

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

This equation gives us a way of solving for the eigenvalues! You might recognize those two quantities that are computed as the trace and determinant of A. This equation is the **characteristic equation**[1].

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

[1] We defined the characteristic equation before, in Chapter 3. We'll see they are the same equation later.

Putting these quantities into the quadratic formula, and defining the discriminant $\Delta = (\text{Tr}(A))^2 - 4\det(A)$, then the eigenvalues are:

$$\lambda = \frac{Tr(A) \pm \sqrt{\Delta}}{2}$$

Just as in Chapter 3, the form of the solution will depend on whether Δ is positive (two real λ), negative (two complex λ) or zero (one real λ). Today, we will **focus on the distinct eigenvalues** case.

Examples: Solve for eigenvalues and eigenvectors

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find eigenvalues and eigenvectors for A.

SOLUTION: We could jump right to the characteristic equation, but for practice its good to write down what it is we actually want to solve (the unknowns below are λ, v_1, v_2):

$$\begin{aligned} 7v_1 + 2v_2 &= \lambda v_1 \\ -4v_1 + v_2 &= \lambda v_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (7-\lambda)v_1 + 2v_2 &= 0 \\ -4v_2 + (1-\lambda)v_2 &= 0 \end{aligned}$$

For this to have a non-zero solution v_1, v_2, the determinant of the coefficient matrix must be zero:

$$(7-\lambda)(1-\lambda) + 8 = 0 \quad \Rightarrow \quad \lambda^2 - 8\lambda + 15 = 0$$

(Note that the trace is 8, determinant is 15). This factors, so we can solve for λ:

$$(\lambda - 5)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 3, 5$$

Now, for each λ, go back to our system of equations for v_1, v_2 and solve. NOTE: By design, these equations should be multiples of each other!

For $\lambda = 3$:

$$\begin{aligned} (7-3)v_1 + 2v_2 &= 0 \\ -4v_2 + (1-3)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4v_1 + 2v_2 &= 0 \\ -4v_2 - 2v_2 &= 0 \end{aligned} \quad \Rightarrow \quad 4v_1 + 2v_2 = 0$$

There are an infinite number of solutions (and there always be). We want to choose "nice" values of v_1, v_2 that satisfies this relationship (or alternatively, lies on the line). An easy way of writing v_1, v_2 is to notice the following:

Given $ax + by = 0$, we can choose $x = b, y = -a$ to lie on the line.

Continuing, we'll take our vector $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. As we'll see in the exercises, we can take any scalar multiple of $\mathbf{v}$ as our answer, so you could have taken $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as well.

Now go through the same process to find the eigenvector for $\lambda = 5$:

$$\begin{aligned} (7-5)v_1 + 2v_2 &= 0 \\ -4v_1 + (1-5)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad 2v_1 + 2v_2 = 0 \quad \Rightarrow \quad v_1 + v_2 = 0$$

Therefore, we take $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

A Little Theory

So far, we've said that, if $\lambda, \mathbf{v}$ form an eigenvalue-eigenvector pair, then $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

As we've now seen, we might have two λ's. How do we deal with that?

- If $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ are eigenvalues and eigenvectors for matrix A, then both

$$\mathbf{x}_1 = \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2 = \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

  are each solutions to $\mathbf{x}' = A\mathbf{x}$.

- If $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ are eigenvalues and eigenvectors for matrix A, the linear combination of these is a solution:

$$C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Lastly, $\mathbf{x}_1, \mathbf{x}_2$ will form a fundamental set of solutions if they are not constant multiples of each other.

- In this section, if λ_1, λ_2 are two distinct real numbers, then the eigenvectors will not be constant multiples of each other so that the linear combination gives the general solution:

$$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

Let's now find the general solution to a system.

Example: Solve a Linear System

Solve the linear system using eigenvectors:

$$\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\mathbf{x}$$

SOLUTION: Compute the eigenvalues and eigenvectors. The determinant is 8, the trace is 6. The characteristic equation is:

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 4) = 0$$

Therefore, $\lambda = 2, 4$.

- For $\lambda = 2$:

$$\begin{array}{rl} (3-2)v_1 + v_2 & = 0 \\ v_1 + (3-2)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} v_1 + v_2 & = 0 \\ v_1 + v_2 & = 0 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- For $\lambda = 4$,

$$\begin{array}{rl} (3-4)v_1 + v_2 & = 0 \\ v_1 + (3-4)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} -v_1 + v_2 & = 0 \\ v_1 - v_2 & = 0 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is:

$$\mathbf{x}(t) = C_1\mathrm{e}^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2\mathrm{e}^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example: Solve the linear system.

The technique is the same as the last example. Write it down, try it out, then come back to this page to see if we have the same answer.

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} \qquad \mathrm{Tr}(A) = 1 \quad \det(A) = -2 \quad \Delta = 9$$

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, or $(\lambda + 1)(\lambda - 2) = 0$.

The eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found by solving the system above. For $\lambda = -1$:

$$\begin{aligned} (3+1)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2+1)v_2 &= 0 \end{aligned} \qquad 2v_1 - v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{aligned} (3-2)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2-2)v_2 &= 0 \end{aligned} \qquad v_1 - 2v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution to the system of differential equations is then:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example:

Consider the second order DE: $y'' + 4y'' + 3y = 0$. In Chapter 3, we looked at something we called the characteristic equation:

$$r^2 + 4r + 3 = 0 \quad \Rightarrow \quad (r+1)(r+3) = 0 \quad \Rightarrow \quad r = -1, -3.$$

and we solved the second order equation. Notice that if we convert this to a system of first order equations, by letting $x_1 = y$ and $x_2 = y'$, then we get:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_1 - 4x_2 \end{aligned} \qquad \Rightarrow \quad \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{x}$$

Now the trace is -4 and the determinant is 3, so the new characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0$$

Which is the same! With $\lambda = -1, -3$, we would still need to find eigenvectors before we would have the solution (try it!).

Summary

To solve the system

$$\begin{aligned} x' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned}$$

the ansatz is $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$. We found that the λ, $\mathbf{v}$ that work are called **eigenvalues** and **eigenvectors** of the coefficient matrix.

To compute $\lambda, \mathbf{v}$, we first write the equations we're solving, we'll call this equation (*):

$$\begin{aligned} (a-\lambda)v_1 \quad &+bv_2 &= 0 \\ cv_1 \quad &+(d-\lambda)v_2 &= 0 \end{aligned}$$

The system has a non-zero solution for v_1, v_2 only if the determinant of the coefficient matrix is zero. This is the characteristic equation, used to solve for the eigenvalues $\lambda_{1,2}$.

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

The solution then depends on the discriminant (as it did in Chapter 3). The first case is when the discriminant is positive so we have two real roots.

If $\mathrm{Tr}(A)^2 - 4\det(A) > 0$, then λ is two distinct, real numbers: $\lambda = r_1, r_2$. For each λ, go back to Equation (*) and solve for an eigenvector. Once those have been determined, the full solution to the DE is given by:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{r_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{r_2 t}\mathbf{v}_2$$

We'll look more at graphing these, and the other cases for the quadratic in future sections.