Study Question Solutions

1. What does it mean to say that something (e.g., a differential equation) is "dimensionless"?

   A quantity is dimensionless if it does not have a dimension (such as mass, length or time) attached to its unit of measure. To say that a differential equation is dimensionless means much the same thing- no mass, length or time is being measured.

2. For the line of best fit, what does "best" mean?

   "Best" means that we are finding the slope and intercept that will minimize the error function. The error function was:

$$E_2(m,b) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

3. Explain the difference between: Deterministic vs Stochastic; Continuous vs Discrete; Empirical vs Analytic.

   Deterministic/Stochastic: Follows from some algebraic formula (like $y = f(x)$), while Stochastic means that we have to use probabilistic descriptions. For example, in a deterministic difference equation, I can compute one exact value for the future state, but in a stochastic model, I must compute a probability of being in a particular future state.

   Continuous/Discrete: Refers to how time is being measured- either continuously (like in a differential equation) or in discrete samples of time (like a difference equation).

   Empirical/Analytic: An empirical model is one that is derived solely from data. An analytic model is one that is derived from the principles that we think govern the object under study. That is, we can explain the data using an empirical model, but it gives no hint as to *why* the data is formed that way.

4. Suppose that $x_{k+1} = Ax_k$ for some real number A.

   If we begin with some initial value, x_0, write x_k in terms of A and x_0 (This is called a closed form for the recurrence relation).

$$\text{Solution: } x_k = A^k x_0$$

   Is there a fixed point if $|A| < 1$? If $|A| > 1$?

   To find the fixed point of $x_{k+1} = f(x_k)$, we must solve $x = f(x)$. In this case, we see that for non-zero A, $x = 0$ will always be a fixed point. The question about the size of A was meant to get you to think about whether or not the orbit of x_0 will converge to the fixed point:

   - If $|A| < 1$, then $|x|^k \to 0$, so 0 will be an attracting fixed point.

- If $|A| > 1$, $|x|^k \to \infty$, so 0 will not be an attracting fixed point.

When will $\sum_{k=0}^{N} x_k$ have a limit as $N \to \infty$? In that case, what is the (infinite) sum?

We can rewrite

$$\sum_{k=0}^{N} x_k = \sum_{k=0}^{N} A^k x_0 = x_0 \sum_{k=0}^{N} A^k = \frac{1 - A^{N+1}}{1 - A}$$

so if $|A| < 1$, the infinite sum is $\frac{1}{1-A}$. If $|A| \geq 1$, the infinite sum diverges.

5. Suppose that some quantity of interest involves the following variables:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

Find a complete set of dimensionless products.

A generic dimensionless product:

$$[\pi] = \frac{L^a}{T^a} \cdot L^b \cdot \frac{L^c}{T^{2c}} \cdot \frac{M^d}{L^{3d}} \cdot \frac{M^e}{L^e T^e}$$

so that:

$$[\pi] = M^{d+e} L^{a+b+c-3d-e} T^{-a-2c-e}$$

We have three equations in five unknowns, therefore two are free. Note that, because of the first equation, we cannot freely choose both d and e for the free variables. But we can choose, for example, c and e and write everything else in terms of these:

$$\begin{aligned} d &= -e \\ a + b - 3d &= -c + e \\ a &= -2c - e \end{aligned}$$

For a complete set, choose $e = 0, c = 1$ and $e = 1, c = 0$. Substitution gives $\{a, b, c, d, e\}$ as (respectively) $\{-2, 1, 1, 0, 0\}$ and $\{-1, -1, 0, -1, 1\}$. This gives the dimensionless variables as:

$$\pi_1 = \frac{rg}{v^2}, \quad \pi_2 = \frac{\mu}{vr\rho}$$

NOTE: You could double check this by making sure that π_1 and π_2 are dimensionless.

6. (Use a calculator) We hypothesize that $y \propto z^{1/2}$. Does the following data support this? If so, find the constant of proportionality:

y	6.4	9.1	11.1	12.8	14.3
z	3	6	9	12	15

Using a calculator, we compare y to $\sqrt{z}$ (this is one method to linearize the proportion):

y	6.4	9.1	11.1	12.8	14.3
$\sqrt{z}$	$\sqrt{3}$	$\sqrt{6}$	3	$2\sqrt{3}$	$\sqrt{15}$

If $y = k\sqrt{z}$, then $k = \frac{y}{\sqrt{z}}$, so by checking these numbers, we see that the data does support this hypothesis, and the $k \approx 3.7$.

(Using Matlab, we could plot the data and compute the line of best fit to determine the constant of proportionality- In this problem, we are just approximating the slope.)

7. If A is proportional to the square of B, how would we linearize the relationship? Same question, but $A = k\mathrm{e}^B$?

   For $A = kB^2$, we could let $x = B^2$ so that $y = kx$. Alternatively, we could take logarithms so that $\ln(A) = \ln(k) + 2\ln(B)$. This is what we do in allometric problems.

   For the second part, we can do the same thing. Either let $x = \mathrm{e}^B$ or take logs so that $\ln(A) - \ln(k) = B$.

8. Prove the two properties for proportions: (i) If $x \propto y$, then $y \propto x$ (ii) If $x \propto y$, $y \propto z$, then $x \propto z$

   For the first, $y \propto x$ means that $y = kx$, $k \neq 0$. Therefore, $x = \frac{1}{k}y$, and $x \propto y$.

   In the second,

$$x = k_1 y, y = k_2 z \Rightarrow x = k_1 k_2 z \Rightarrow x = kz$$

9. Suppose that we assume that Supply and Demand curves are linear. Here is some hypothetical data (these are relative quantities, so we can have negative prices and quantities). Fill in the missing numbers.

Quantity	Price
3	2
-1	2
-1	14/3
5/3	14/3
1.67	2.89
-0.11	2.89

   Find the equilibrium state.

   From the data, we see that the supply line is $y = x + 3$, and the demand line is $y = -\frac{2}{3}x + 4$. The point of intersection is the equilibrium, $(0.6, 3.6)$. It looks like the market adjustments are moving toward equilibrium.

10. Assume that heat loss is proportional to surface area. How much more heat loss will a 12 inch cube have than a 6 inch cube? *NOTE: In this problem, we will take "6 inch cube" to mean that each edge is 6 inches long*

    The surface area of the small cube is $6 \cdot 36$. Scaling the sides by k means the new surface area is $6 \cdot (6k \cdot 6k) = k^2(6 \cdot 36)$. Therefore, double the lengths increases the surface area 4 times (and so there will also be 4 times as much heat loss).

11. Describe Newton's Method graphically and give the formula.

    We'll do the graph in-class. The formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

12. For spheres and geometrically similar boxes, show that surface area, S, is proportional to volume V to the 2/3 power.

    For spheres, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. From the first, $r = \sqrt[3]{\frac{3}{4\pi}} V^{1/3} = kV^{1/3}$. Inserting this into the second gives:

$$S = 4\pi k^2 V^{2/3} = k_1 V^{2/3}$$

    For boxes, use a "template" box with side length 1, width w and height h (these values are fixed, and therefore constant, for the class of boxes).

    Now, given any other box in the class, the length is k, width is kw and height is kh. Therefore,

$$S = k^2(wh + w + h) = c_1 k^2 \quad V = k^3 wh = c_2 k^3$$

    Therefore $k = \frac{1}{c_2^{1/3}} V^{1/3}$, and

$$S = \frac{c_1}{c_2^{2/3}} V^{2/3} = c_3 V^{2/3}$$

13. Suppose that gymnasts are geometrically similar with a characteristic scaling k. First, write each of the following assumptions as mathematical statements:

    (a) Ability is proportional to strength, and inversely proportional to weight.

$$A = k_1 \frac{S}{W} \text{ or } A \propto \frac{S}{W}$$

    (b) Strength is proportional to muscle area.

$$S = k_2 M \text{ or } S \propto M$$

Next, how should muscle area scale with k? (Hint: Think of muscle area as proportional to surface area) How should weight scale with k? Put the 4 statements together to show that $A \propto \frac{1}{k}$.

$$M \propto k^2, \quad W \propto k^3$$

We need to relate both S and W to k. We have W, so for S,

$$S \propto M \propto k^2$$

Now,

$$A \propto \frac{S}{W} \propto \frac{k^2}{k^3} = \frac{1}{k}$$

Therefore, gymnasts should be short (this line of reasoning showed that ability is inversely proportional to height).

14. Find the volume flow rate of blood flowing in an artery, $\frac{dV}{dt}$, as a function of the pressure drop per unit length, P, the radius r, the density ρ and the viscosity μ.

Here is a list of dimensions. Fill in the blanks:

Variable	dV/dt	P	r	ρ	μ
Dimension		$M/(LT^2)$	L	M/L^3	$M/(LT)$

Now perform dimensional analysis and apply Buckingham's π Theorem.

SOLUTION: A generic dimensionless variable will have the form:

$$\pi = (dV)^a P^b r^c \rho^d \mu^e$$

so that, in dimensions,

$$1 = \frac{L^{3a}}{T^a} \cdot \frac{M^b}{L^b T^{2b}} \cdot L^c \cdot \frac{M^d}{L^{3d}} \cdot \frac{M^e}{L^e T^e}$$

and by equating exponents (L,M, then T):

$$\begin{array}{cccccc} 3a & -b & +c & -3d & -e & =0 \\ & b & & +d & +e & =0 \\ -a & -2b & & & -e & =0 \end{array}$$

We have three equations and five unknowns, so we can choose the two free variables. Since we're going to want to solve for dV/dt, we'll choose a to be one of the free ones. Arbitrarily, we choose e as the second.

There are several ways of writing the solutions. But notice that, given the choice we made for free variables, we can write:

$$\begin{aligned} -b + c - 3d &= -3a + e \\ b + d &= -e \\ -2b &= a + e \end{aligned}$$

Which we could leave as is, or we can go ahead and solve, getting:

$$b = -\frac{1}{2}a - \frac{1}{2}e, \quad c = -2a - e, \quad d = \frac{1}{2}a - \frac{1}{2}e$$

Substituting $a = 2, e = 0$ and $a = 0, e = 2$, we get:

$a = 2, b = -1, c = -4, d = 1, e = 0$ and $a = 0, b = -1, c = -2, d = -1, e = 2$

Therefore,

$$\pi_1 = \left(\frac{dV}{dt}\right)^2 \frac{\rho}{Pr^4}, \quad \pi_2 = \frac{\mu^2}{Pr^2\rho}$$

By the Buckingham π Theorem, there exists an f so that:

$$f\left(\left(\frac{dV}{dt}\right)^2 \frac{\rho}{Pr^4}, \frac{\mu^2}{Pr^2\rho}\right) = 0$$

By the Implicit Function Theorem, we assume that we can solve this so that there exists a function h:

$$\left(\frac{dV}{dt}\right)^2 \frac{\rho}{Pr^4} = h\left(\frac{\mu^2}{Pr^2\rho}\right)$$

so that:

$$\frac{dV}{dt} = r^2 \cdot \sqrt{\frac{P}{\rho}} \cdot h\left(\frac{\mu^2}{Pr^2\rho}\right)$$

15. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q. Use dimensional analysis to determine an equation for power. Below are listed the dimensions:

Variable	P	w	h	q
Dimension	ML^2T^{-3}	$ML^{-2}T^{-2}$	L	L^3T^{-1}

Set up the equations from:

$$\left(\frac{ML^2}{T^3}\right)^a \left(\frac{M}{L^2T^2}\right)^b L^c \left(\frac{L^3}{T}\right)^d$$

From which:

$$\begin{aligned} a + b &= 0 \\ -3a - 2b - d &= 0 \\ 2a - 2b + c + 3d &= 0 \end{aligned}$$

Take a as the free variable, and set it to 1, then $b = c = d = -1$. This gives one dimensionless variable,

$$\pi_1 = \frac{P}{whq}$$

so that the Buckingham π Theorem states that there is a function f so that

$$f\left(\frac{P}{whq}\right) = 0$$

Since $f(\pi) = 0$, π is a constant. Thus,

$$P \propto whq \text{ or } P = kwhq$$

16. What is a "fixed point" for a difference equation of the form $x_{k+1} = f(x_k)$?

    The fixed point is the value of x so that $x = f(x)$.

    Find the fixed point(s) if $x_{k+1} = kx_k(1 - x_k)$, k is a real number.

    In this case,

    $$x = kx(1-x) \Rightarrow kx^2 - kx + x = 0 \Rightarrow x(kx + (1-k)) = 0$$

    so that $x = 0$ or $x = \dfrac{k-1}{k}$

17. State the Buckingham π Theorem. An equation is dimensionally homogeneous (or dimensionally consistent) if and only if it can be put into the form:

    $$f(\pi_1, \pi_2, \ldots, \pi_n) = 0$$

    where $\pi_1, \pi_2, \ldots, \pi_n$ form a complete set of dimensionless products.

18. What is the justification for saying that: "If $f(x, y) = 0$, then we can solve for y as a function of x: $y = g(x)$". Apply this to the equation for a unit circle. Using the unit circle, can I always solve for y in any small region about any fixed (x^*, y^*) in the domain?

    The justification is the Implicit Function Theorem (you do not have to know what it is, just the name for now). For example, from the equation for a circle (in the form given from the theorem):

    $$x^2 + y^2 - 1 = 0 \Rightarrow y = \pm\sqrt{1 - x^2}$$

    where we would have to choose a $+$ or $-$. We cannot do this locally about any point, since the two points $(-1, 0)$ and $(1, 0)$ have vertical tangent lines, and a local region about these points would necessarily have to include points from both the upper and lower half circle.

19. Draw a scatter plot with several data points and draw your estimate of the line of best fit. Recall that we defined our data as (x_i, y_i), with $\hat{y}_i = mx_i + b$ and $y_i = \hat{y}_i + \epsilon_i$. On the plot, show a sample of the following quantities:

    (a) The horizontal line at $\hat{y}$

    (b) $y_i - \overline{y}$ (This is used in computing total variation).

(c) $y_i - \hat{y}_i$ (This is called the variation that is NOT explained by the model)

(d) $\hat{y}_i - \overline{y}$ (This is called the variation that IS explained by the model)

We'll answer this question in-class.

20. The crucial element of our modeling of biological systems came from our relationship of surface area to mass, $S \propto M^{2/3}$. How did we arrive at this?

    See Question 12. In this case, if two things are geometrically similar, and S and V are the surface area and volume of the template for the class, then for a particular member of the class, its surface area $\hat{S}$ and volume $\hat{V}$ are related by:

    $$\hat{S} \propto k^2 S, \quad \hat{V} \propto k^3 V$$

    where k is the scaling factor (this assumes that the organism is three dimensional).

21. Is it possible that within any single group, we can have an affine[1] relationship with a slope of $-1/3$, but when we put the groups together, we get a relationship with slope $-1/4$? If this is true, should our model have used $-1/3$ or $-1/4$? Discuss.

    This is true (we had some data with exactly those qualities). The second part of the question gets at the heart of what we are trying to model- Our model was looking at scaling over all kinds of *different* species. But, this question remains as a big unanswered question in biology.

22. To understand how branchings scale, let us consider the following model. Take an interval, $[0, 1]$. We'll say that the first "branching" cuts the interval into two pieces, each of equal length. The second branching cuts each of those into two pieces of equal length, and so on. The following table may help. Fill in the last entry:

Branching	Number of Intervals	Width of each
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
$\vdots$	$\vdots$	$\vdots$
k	2^k	$\frac{1}{2^k}$

Suppose we require that the last branch cover a set width, say $\frac{1}{2^{10}}$. How many branchings are required?

$$\frac{1}{2^k} = \frac{1}{2^{10}} \Rightarrow k = 10$$

[1]Affine refers to $y = mx + b$ rather than $y = mx$, which is linear.

Do the same problem, except take the initial interval to have length S.

In this case, at stage k we cover subintervals of length $\frac{S}{2^k}$. In this case,

$$\frac{S}{2^k} = \frac{1}{2^{10}} \Rightarrow k = \log_2(S) + 10$$

Using your previous answer, if length S is multiplied by $2^7 = 128$, how much larger is k?

$$k = \log_2(2^7 S) + 10 = 7 + (\log_2(S) + 10)$$

If length S is multiplied by c, how much larger is k?

We would add $\log_2(c)$. Note that this grows very slowly compared to c.

NOTE: It might be interesting at some point to see how this would work using volumes instead of intervals, with d branchings instead of 2.

23. We said that, if we have two data sets with n points, x and y, and if they both have zero mean, then the error function (sum of squares) is a function of one variable (model: $y = mx$). Write down the error function and find the slope that minimizes the error.

If the data in x and y have zero mean, then

$$E(m) = \sum_{i=1}^{n} (y_i - mx_i)^2$$

To minimize, set the derivative to zero and solve for m:

$$\frac{dE}{dm} = 2\sum_{i=1}^{n} (y_i - mx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} (-x_i y_i + mx_i^2) = 0 \quad \Rightarrow \quad m\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

You could solve for m now, but we'll continue so that the solution looks like that we found in class: Put back the means, and

$$m\sum_{i=1}^{n} (x_i - \overline{x})^2 = \sum_{i=1}^{n} (x_i - \overline{x})y_i \quad \Rightarrow \quad mS_x^2 = S_{xy}^2$$

so that

$$m = \frac{S_{xy}^2}{S_x^2}$$

What is significant here is that this formula works when x, y are NOT perfectly linearly related (as in Problem 25 below).

24. Defining $y_i = mx_i + b + \epsilon_i$, take the mean of both sides to find a formula for b in terms of $m, \overline{x}, \overline{y}$. You may assume that the residuals have zero mean.

$$\frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(mx_i + b + \epsilon_i) = m \cdot \frac{1}{n}\sum_{i=1}^{n} x_i + b \cdot \frac{1}{n} \cdot n + \frac{1}{n}\sum_{i=1}^{n} \epsilon_i$$

Now, we see that $\overline{y} = m\overline{x} + b + 0$, or $b = \overline{y} - m\overline{x}$

25. Show, by using the definition of the covariance, that if we assume $y_i = mx_i + b$, then $S_{xy}^2 = mS_x^2$. Did we need to assume a perfect linear fit for this formula to work?

(Compare to Problem 23). In assuming $y_i = mx_i + b$,

$$S_{xy}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(mx_i + b - (m\overline{x} + b)) =$$

$$m\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = mS_x^2$$

26. Show, by using the definition of the correlation, that if we assume $y_i = mx_i + b$, then $r_{xy} = \pm 1$.

$$r_{xy} = \frac{S_{xy}^2}{S_x S_y} = \frac{mS_x^2}{S_x S_y}$$

Let's recall the relationship between S_x and S_y if we assume that $y_i = mx_i + b$:

$$S_y^2 = \frac{1}{n-1}\sum + i = 1^n (y_i - \overline{y})^2 = \frac{1}{n-1}((mx_i + b) - (m\overline{x} + b))^2 =$$

$$m^2\frac{1}{n-1}\sum + i = 1^n (x_i - \overline{x})^2 = m^2 S_x^2$$

so $S_y = |m|S_x$. Put this back into our expression:

$$r_{xy} = \frac{mS_x^2}{S_x S_y} = \frac{mS_x^2}{S_x|m|S_x} = \frac{m}{|m|} = \pm 1$$

27. Can the variance produce any real number? Can the covariance? (Give algebraic reasons). The variance is a sum of squares, so it cannot produce a negative number. On the other hand, the covariance can be written as a dot product, which can produce any real number.

28. Show, using dot products and norms, that the correlation must be a number between ± 1.

As in class, assume the vectors $\mathbf{x}, \mathbf{y}$ are formed from the mean-subtracted data. Then

$$S_{xy}^2 = \frac{1}{n-1}\mathbf{x}\cdot\mathbf{y} = \frac{1}{n-1}\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)$$

Now,

$$r_{xy} = \frac{S_{xy}^2}{S_x S_y} = \frac{\frac{1}{n-1}\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)}{\frac{1}{\sqrt{n-1}}\|\mathbf{x}\|\frac{1}{\sqrt{n-1}}\|\mathbf{y}\|} = \cos(\theta)$$

29. Suppose our data produces the following summary statistics:

    (a) $S_x^2 = 3$
    (b) $S_{xy}^2 = -7$
    (c) $\overline{x} = 2$
    (d) $\overline{y} = 3$

    Find the line of best best.

    We put down the following theorem in class (which we also proved in this review):

    Theorem: The line of best fit is given by slope m, and intercept b:

$$m = \frac{S_{xy}^2}{S_x^2} \qquad b = \overline{y} - m\overline{x}$$

    So in this case,

$$m = \frac{-7}{3}, \quad b = 3 + \frac{7}{3}\cdot 2 = \frac{23}{3}$$

    so the line of best fit is:

$$y = \frac{-7}{3}x + \frac{23}{3}$$