Solutions: Review Questions

1. The solution is:

$$\boldsymbol{x} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

2. If AB is invertible, then by the IMT, there is a W such that $(AB)W = I$. By the rules of matrix multiplication, $A(BW) = I$. Now, by the IMT, since A is square and there is a C such that $AC = I$, A is invertible.

3. If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly dependent, then there are infinitely many (nontrivial) solutions to the equation:

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = 0$$

Apply the linear transformation T to both sides, and simplify. This shows that any solution to the equation above is also a solution to:

$$c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + c_3T(\boldsymbol{v}_3) = 0$$

which says that $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ form a linearly dependent set.

4. If $H\boldsymbol{x} = \boldsymbol{v}$ is consistent for all $\boldsymbol{v}$, then every row of H has a pivot. Therefore, there are 7 pivots. Since H has only 7 columns, there must be a pivot in every column as well. Therefore, there are no free variables, and therefore, the solution must be unique.

5. We can show the mapping is nonlinear in a couple of ways. For example, $T(0, 0) \neq (0, 0)$

6. Suppose that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent. Let V be a matrix whose columns are the $\boldsymbol{v}_i$'s. Then: $V\boldsymbol{c} = \mathbf{0}$ has only the trivial solution. Now the question is whether or not $AV\boldsymbol{c} = \mathbf{0}$ has only the trivial solution. The answer is yes. Because A is invertible, we can rewrite the previous equation as $A^{-1}AV\boldsymbol{c} = A^{-1}\mathbf{0}$. Now this has only the trivial solution, therefore so does $AV\boldsymbol{c} = \mathbf{0}$.

7. False. The description given is for a *function*; that is, each x has only one y associated with it. For a function to be 1-1, each y must have come from a unique x,

$$x_1 \neq x_2 \Rightarrow T(x_1) \neq T(x_2)$$

8. You can ignore the matrix B, although it is one of the matrices C. We are looking for the columns of C,

$$A\mathbf{c}_1 = \mathbf{e}_1, \quad A\mathbf{c}_2 = \mathbf{e}_2, \quad A\mathbf{c}_3 = \mathbf{e}_3$$

To solve these at one time, we will row reduce the augmented matrix $[A \mid I_2]$:

$$\left[\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & -1 \end{array}\right]$$

The interpretation is (by column):

$$x_1 = -x_3, x_2 = 1 + 2x_3, x_3 = x_3(\text{free}) \Rightarrow \mathbf{c}_1 = (0, 1, 0)^T + x_3(-1, 2, 1)^T$$

and

$$\hat{x}_1 = 1 - \hat{x}_3, \hat{x}_2 = -1 + 2\hat{x}_3, \hat{x}_3 = \hat{x}_3 \Rightarrow \mathbf{c}_2 = (1, -1, 0)^T + \hat{x}_3(-1, 2, 1)$$

Now choose some arbitrary numbers for x_3 and $\hat{x}_3$ to find three different C's- Note that C will be 3×2

9. For a unique solution, $h \neq 6$, and k is anything. For infinitely many solutions, $h = 6$ and $k = 2$. For no solution, $h = 6$ and $k \neq 2$.

10. Let us say that the coefficient matrix is $m \times n$. If the system is underdetermined, then $m < n$. Therefore, at best, we can have m pivots, which leaves at least $n - m$ free variables. Therefore, if the system is consistent, it must have an infinite number of solutions.

11. Note that in each case, E needs to be 3×3 for EA to be defined.

$$E = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

12. Asking if the set spans $\mathbb{R}^n$ means that we are checking if every vector in $\mathbb{R}^n$ can be written as a linear combination of those vectors. If we put the vectors in the matrix A, we are asking if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$. Once we take the RREF of A, we then check to see if there is a pivot in every row. (Recall that a "pivot in every column" is for linear independence).

13. This question is basically the same as before, except we are asking for a solution *for one value* of $\boldsymbol{b}$.

14. (a) $\boldsymbol{y}_1, \boldsymbol{y}_2$ must be in $\mathbb{R}^3$, and $\boldsymbol{x}_1, \boldsymbol{x}_2$ must be in $\mathbb{R}^4$. (b) $A\boldsymbol{x} = \boldsymbol{w}$ has a solution. By the matrix multiplication properties, if $\boldsymbol{w} = \boldsymbol{y}_1 + \boldsymbol{y}_2$, then $\boldsymbol{x}$ must be $\boldsymbol{x}_1 + \boldsymbol{x}_2$.

15. The two solutions are (in order):

$$\boldsymbol{x} = r \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The latter solution is a translated version of the former solution.

16. This is true. It says that the two vectors are constant multiplies of each other.

17. Lots of examples. For instance, let $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}_4 = (1, 1, 1, 1)^T$. Let $\boldsymbol{v}_3 = (0, 1, 1, 1)^T$.

18. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we need to satisfy the following sets of equations:

$$\begin{aligned} a + 2b &= -3 \\ c + 2d &= 0 \\ -2a + b &= 1 \\ -2c + d &= 2 \end{aligned}$$

We have a variety of ways to solve this. $a = -1, b = -1, c = -\frac{4}{5}, d = \frac{2}{5}$

19. If $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution, there must be a pivot in every column. In this case, there are n. But since A is $n \times n$, this means that there must be a pivot in every row as well. Therefore, the columns do span $\mathbb{R}^n$. In general, this is not true- That is, in general, having a pivot in every column does not imply that there is a pivot in every row.

20. See Problem 2, and let $B = A$.

21.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 13 & 7 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

22. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$, there must be a pivot in every row. Since A is square, this implies that there must be a pivot in every column. Therefore, the columns of A also span $\mathbb{R}^n$. This is not true in general.

23. False. Because $[A|I] \rightarrow [I|A^{-1}]$, the row operations that reduce A to I also "reduce" I to A^{-1}.

24. The first and second columns of AB are: $(-13, 32)^T$ and $(20, -49)^T$. The $(3, 1)$ entry of $B^T A$ is -7. We can't find the inverse of $B^T B$ because it is not row reducible to the identity (we get one free variable, x_3).

25. Suppose $\mathbf{x}$ is a vector satisfying $A\mathbf{x} = \mathbf{0}$. We will show that this implies that $\mathbf{x} = 0$, so the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution.

$$\begin{array}{ll} A\mathbf{x} = \mathbf{0} & \text{Given} \\ CA\mathbf{x} = C\mathbf{0} = \mathbf{0} & \text{Multiply by } C \\ I_n\mathbf{x} = \mathbf{0} & \text{Since } CA = I_n \\ \mathbf{x} = \mathbf{0} & \end{array}$$

We have shown that, if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{x} = 0$ is the only solution to $A\mathbf{x} = 0$.

26. The general matrix is $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, so the matrix is this case is: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

27. $T(\boldsymbol{x}) = [3 \quad -4 \quad 0 \quad 8]\boldsymbol{x}$

28. Show that

- $T_1(T_2(\boldsymbol{u}+\boldsymbol{v})) = T_1(T_2(\boldsymbol{u})) + T_1(T_2(\boldsymbol{v}))$:

$$T_1(T_2(\boldsymbol{u}+\boldsymbol{v})) = T_1(T_2(\boldsymbol{u}) + T_2(\boldsymbol{v})) \text{ by the linearity of } T_2$$

$$T_1(T_2(\boldsymbol{u}) + T_2(\boldsymbol{v})) = T_1(T_2(\boldsymbol{u})) + T_1(T_2(\boldsymbol{v})) \text{ by the linearity of } T_1$$

- $T_1(T_2(a\boldsymbol{u})) = aT_1(T_2(\boldsymbol{u}))$

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29. The matrix is: $\begin{pmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{pmatrix}$

30. Note that we can answer these questions without row reduction:

To see if T is 1-1, there can be at MOST one solution to $A\boldsymbol{x} = \boldsymbol{b}$ for each $\boldsymbol{b}$. This means that there cannot be free variables, and this in turn means that there must be a pivot in every column. Since there are only two columns, and they are not constant multiples of each other, we can conclude that there will be a pivot in each column, and therefore, T will be 1-1.

Since A is 3×2, we can have at most 2 pivots (and indeed, we do have two pivots). Therefore, there cannot be a pivot in every row, and that means that the columns will not span $\mathbb{R}^3$. Therefore, the function T is not onto $\mathbb{R}^3$.

31.

$$I - BAB^{-1} = C \Rightarrow I - C = BAB^{-1} \Rightarrow B^{-1}I - B^{-1}C = AB^{-1} \Rightarrow B^{-1}B - B^{-1}CB = A \Rightarrow I - B^{-1}CB = A$$

32. (a) If the columns of A are linearly independent, then the only solution to:

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \ldots + c_n\boldsymbol{a}_n = \mathbf{0}$$

is $c_1 = c_2 = \ldots = c_n = 0$ (this is the definition of linear independence).

(b) Rewrite the above equation in Matrix-Vector form:

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \ldots + c_n\boldsymbol{a}_n \Leftrightarrow [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \ldots \boldsymbol{a}_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \Leftrightarrow A\boldsymbol{c}$$

(c) From (a) and (b), the only solution to

$$A\mathbf{c} = \mathbf{0} \text{ is } \boldsymbol{c} = \mathbf{0}$$