

## MATH 300, Second Exam REVIEW SOLUTIONS

*NOTE:* You may use a calculator for this exam- You only need something that will perform basic arithmetic.

1. Let  $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ , Let  $S$  be the parallelogram with vertices at  $\mathbf{0}$ ,  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{u} + \mathbf{v}$ . Compute the area of  $S$ .

SOLUTION: Take the determinant (absolute value if you reverse the columns)- The area is 4.

2. Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ ,  $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$ , and  $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$ .

If  $\det(A) = 5$ , find  $\det(B)$ ,  $\det(C)$ ,  $\det(BC)$ .

SOLUTION:

To get  $B$  from  $A$ , we take  $2r_3 + r_1 \rightarrow r_1$  and  $3r_3 + r_2 \rightarrow r_2$ . These row operations do not change the determinant, so  $\det(A) = \det(B) = 5$ .

To get the determinant of  $C$ , multiply the determinant of  $A$  by  $-2$  (negative because of the row swap); we get  $-10$ .

The determinant of  $BC$  is then  $5 \cdot -10 = -50$

3. Assume that  $A$  and  $B$  are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**TYPO:**  $B(1,6)$  should be  $-3$  instead of  $3$ . Corrected above.

- (a) Find a basis for  $\text{Col}(A)$ : From the row-reduced matrix, we see that we should use columns 1, 2 and 4. Be sure to use the columns from the original matrix  $A$ ! The basis is the set

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}$$

*Important side note:* The reduced form of  $A$  also says that the third column of  $A$  can be formed by  $4\mathbf{a}_1 - 3\mathbf{a}_2$ , and column 5 is  $-3\mathbf{a}_1 + 5\mathbf{a}_2 - 4\mathbf{a}_4$  (Check it out!)

- (b) Find a basis for  $\text{Row}(A)$ : We can use the reduced rows (written as columns) for the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(c) Find a basis for  $\text{Null}(A)$ : Solve  $A\mathbf{x} = \mathbf{0}$ , and the vectors are the basis vectors:

$$\begin{aligned} x_1 &= -4x_3 + 3x_5 \\ x_2 &= 3x_3 - 5x_5 \\ x_3 &= x_3 \\ x_4 &= 4x_5 \\ x_5 &= x_5 \end{aligned} \Rightarrow \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

4. Determine if the following sets are subspaces of  $V$ . Justify your answers.

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0, b \geq 0, c \geq 0 \right\}, \quad V = \mathbb{R}^3$$

SOLUTION: This set is not closed under scalar multiplication (for example, if  $c = -1$ ).

$$H = \left\{ \begin{bmatrix} a + 3b \\ a - b \\ 2a + b \\ 4a \end{bmatrix}, a, b \text{ in } \mathbb{R} \right\}, \quad V = \mathbb{R}^4$$

SOLUTION: This is the span of  $[1, 1, 2, 4]^T$  and  $[3, -1, 1, 0]^T$ , so  $H$  is a subspace.

$$H = \{f : f'(x) = f(x)\}, V = C^1[\mathbb{R}]$$

( $C^1$  is the space of differentiable functions where the derivative is continuous).

SOLUTION: We could show this directly:

- The zero function is in  $H$  since the derivative of 0 is 0.
- If  $u, v$  are in  $H$ , then  $u' = u$  and  $v' = v$ . Therefore,  $(u + v)' = u' + v' = u + v$ , so  $u + v$  is in  $H$ .
- If  $u \in H$ , then  $u' = u$ . Therefore,  $(cu)' = cu' = cu$ , so  $cu$  is in  $H$ .

$H$  is the set of vectors in  $\mathbb{R}^3$  whose first entry is the sum of the second and third entries,  $V = \mathbb{R}^3$ .

SOLUTION: Rewriting  $H$  algebraically, it is the set of vectors in  $\mathbb{R}^3$  so that

$$\begin{bmatrix} a + b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Since  $H$  is the span of two vectors, it is a subspace.

5. Prove that, if  $T : V \mapsto W$  is a linear transformation between vector spaces  $V$  and  $W$ , then the range of  $T$  is a subspace of  $W$ .

(Done in class)

6. Let  $A$  be an  $n \times n$  matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement “ $A$  is invertible”. Use the following concepts, one in each statement:

SOLUTION:

(a)  $\text{Null}(A) = \{0\}$  (b) Basis is the columns of  $A$  (c) Determinant is not zero

7. If  $A, B$  are  $4 \times 4$  matrices with  $\det(A) = 2$  and  $\det(B) = -3$ , what is the determinant of the following (if you can compute it):

SOLUTIONS:

(a)  $\det(AB) = -6$ , (b)  $\det(A^{-1}) = 1/2$ , (c)  $\det(5B) = 5^3(-3)$ , (d)  $\det(3A - 2B)$  is unknown with what is given, (e)  $\det(B^T) = -3$

8. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

SOLUTION:

No. If we have 10 equations in 12 variables, we must have at least two free variables, and so the null space of the corresponding matrix is the span of at least two linearly independent (nonzero) vectors.

9. Show that  $\{1, 2t, -2 + 4t^2\}$  is a basis for  $\mathbb{P}_2(t)$ . (Hint for the span: Show that an arbitrary polynomial  $a + bt + ct^2$  can be written as a linear combination of the given vectors.)

SOLUTION: (Note that  $P_3$  should have been  $\mathbb{P}_2(t)$ , and is corrected here).

*Side remark:* If a polynomial is equal to another polynomial for all  $t$ , then each of the coefficients must be equal.

Continuing, take  $c_1p_1 + c_2p_2 + c_3p_3 = 0$  and solve:

$$c_1 \cdot 1 + c_2 \cdot 2t + c_3(-2 + 4t^2) = 0 \text{ for all } t$$

This implies that the coefficient of each monomial is zero:

$$\begin{array}{ll} \text{Constants:} & c_1 - 2c_3 = 0 \\ t \text{ terms:} & 2c_2 = 0 \\ t^2 \text{ terms:} & 4c_3 = 0 \end{array}$$

Therefore,  $c_3 = 0$ ,  $c_2 = 0$  which implies that  $c_1 = 0$ . We have shown that the trivial solution is the only solution to  $c_1p_1 + c_2p_2 + c_3p_3 = 0$ , so these vectors are linearly independent.

Do they span? If so, we would have to solve the following for an arbitrary  $a, b, c$ :

$$\begin{array}{ll} \text{Constants:} & c_1 - 2c_3 = a \\ t \text{ terms:} & 2c_2 = b \\ t^2 \text{ terms:} & 4c_3 = c \end{array} \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & 4 & c \end{array} \right]$$

We see that the matrix equation always has a solution. Therefore,  $p_1, p_2$  and  $p_3$  span  $\mathbb{P}_2(t)$ .

Because the vectors are linearly independent and span  $\mathbb{P}_2(t)$ , they form a basis for it.

10. Let  $T : V \rightarrow W$  be a linear transformation on vector space  $V$  to vector space  $W$ . Show that if  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  are linearly independent vectors in  $V$ , then  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$  are linearly independent vectors in  $W$ .

**NOTE:** Rephrase the question like we did in class to:

Let  $T : V \rightarrow W$  be a linear transformation on vector space  $V$  to vector space  $W$ . Show that if  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$  are linearly dependent vectors in  $W$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  are linearly dependent vectors in  $V$ . You will need to assume that  $T$  is 1 - 1.

Proof: if  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$  are linearly dependent, then there is a nontrivial solution to

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = 0$$

Since  $T$  is linear, we can write this equation as:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(0)$$

If  $T$  is 1 - 1, we can say that  $T(a) = T(b)$  implies that  $a = b$ . In this case, we then can say that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

So that we again get a non-trivial solution, and these vectors are linearly dependent.

11. (a) The functions  $f(t) = t$  and  $g(t) = t$  are linearly dependent. Are  $f(t) = t$  and  $g(t) = -t$  linearly dependent?

SOLUTION: Yes, since they are constant multiples of each other.

- (b) Are the functions  $f(t) = t$  and  $g(t) = |t|$  linearly independent or dependent?

SOLUTION: It depends on the interval. If  $t \geq 0$ , the answer is that the functions are linearly dependent. If  $t \leq 0$ , the functions are linearly dependent. However, if the interval contains both positive and negative values, then the functions are linearly *independent*, since they are NOT constant multiples of each other.

*Side Remark:* We included this question to highlight the importance of the interval on which  $t$  is defined when determining if a set of functions is linearly independent. If the domain is not specified, assume the domain is the natural domain.

12. Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x_1 + x_2 &= 7 \\ -3x_1 + x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

We'll need some determinants:

$$\begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 \quad \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 6 \quad \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 16 \quad \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = -14$$

So the solution is:

$$x_1 = \frac{6}{4} = \frac{3}{2} \quad x_2 = \frac{16}{4} = 4 \quad x_3 = \frac{-14}{4} = \frac{-7}{2}$$

13. Let  $A$  be a  $3 \times 4$  matrix. Construct the elementary matrix  $E$  to perform the given row operations, and give the determinant of  $E$ :

(a)  $3r_2 + r_1 \rightarrow r_1$   $E = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $\det(E)=1$

(b)  $r_3 \rightarrow -3r_3$   $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ ,  $\det(E)=-3$

(c) The inverse of the row operation  $2r_1 + r_2 \rightarrow r_2$ .  $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $\det(E)=1$

14. Let  $A, B$  be square matrices. Show that if  $AB$  is invertible, then so is  $A$ . (Hint: If  $AB$  is invertible, let  $W$  be its inverse so that  $ABW = I$ .)

SOLUTION: If  $AB$  is invertible, then there is a matrix  $W$  so that

$$ABW = I \quad \Rightarrow \quad A(BW) = I$$

Therefore, there is a matrix  $C = BW$  such that  $AC = I$ , and since  $A$  is square, the invertible matrix theorem applies, and  $A$  is invertible.

15. Let  $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ , and  $\mathbf{w} = [2, 1]^T$ . Is  $\mathbf{w}$  in the column space of  $A$ ? Is it in the null space of  $A$ ?

SOLUTION: The first question is asking if  $\mathbf{w}$  is in the span of the columns of  $A$ . By inspection, the weights are  $-1/3$  and  $0$ , so yes.

The second question is asking if  $A\mathbf{w} = \mathbf{0}$ . In this case, it is true as well.

*Side Note:* It was only a coincidence that this vector was in both the column space as well as the null space. If  $A$  was  $m \times n$ , then the column space would be in  $\mathbb{R}^m$  and the null space would be in  $\mathbb{R}^n$ , so these would definitely be different.

16. Prove directly (by showing the three conditions hold) that the column space of a matrix is a subspace.

SOLUTION: Notice that a vector  $\mathbf{b}$  is in the column space of  $A$  if and only if  $\mathbf{b} = A\mathbf{x}$  for some  $\mathbf{x}$ . Therefore, this proof follows the proof that the range of a linear transformation is a subspace:

(a)  $\mathbf{0} \in \text{Col}(A)$ , since  $A\mathbf{0} = \mathbf{0}$ .

(b) Let  $\mathbf{b}_1, \mathbf{b}_2 \in \text{Col}(A)$ . Then there exists  $\mathbf{x}_1, \mathbf{x}_2$  so that

$$A(\mathbf{x}_1) = \mathbf{b}_1 \quad A(\mathbf{x}_2) = \mathbf{b}_2$$

Therefore,

$$\mathbf{b}_1 + \mathbf{b}_2 = A(\mathbf{x}_1) + A(\mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2)$$

Therefore,  $\mathbf{b}_1 + \mathbf{b}_2$  is in the column space of  $A$ .

(c) Let  $\mathbf{b} \in \text{Col}(A)$ . Then  $A\mathbf{x} = \mathbf{b}$  for some  $\mathbf{x}$  in the domain, and

$$c\mathbf{b} = cA\mathbf{x} = A(c\mathbf{x})$$

Therefore,  $c\mathbf{b}$  is in the column space of  $A$  for all constants  $c$ .

17. Let  $A$  and  $B$  be given below. (a) Find only the first column of  $AB$ . (b) Find only the  $(3, 1)$  entry of  $B^T A$ . (c) Determine the inverse of  $BB^T$ , if possible.

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

SOLUTION:

- (a) The first column of  $AB$  is  $A\mathbf{b}_1$ , or  $[-13, 32]^T$   
(b) The  $(3, 1)$  entry is the dot product of the third column of  $B$  (third row of  $B^T$ ) with the first column of  $A$ , or 5.  
(c) No shortcuts:  $BB^T = \begin{bmatrix} 6 & -27 \\ -27 & 126 \end{bmatrix}$ , so the inverse is (shortcut for this):

$$\frac{1}{27} \begin{bmatrix} 126 & 27 \\ 27 & 6 \end{bmatrix}$$

18. Let  $A$  be  $m \times n$ . Suppose that for some matrix  $C$ ,  $CA = I$ . Show that the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

SOLUTION: If  $A\mathbf{x} = \mathbf{0}$ , multiply both sides by  $C$  to get  $CA\mathbf{x} = \mathbf{0}$ , or  $\mathbf{x} = \mathbf{0}$ .

19. True or False, and give a short reason:

- (a) If  $AB = I$ , then  $A$  is invertible.  
FALSE. True if  $A$  is square.  
(b) If  $BC = BD$ , then  $C = D$ .  
FALSE. True if  $B$  is invertible.  
(c) If  $A, B$  are  $n \times n$ , then  $(A + B)(A - B) = A^2 - B^2$ .  
FALSE. True if  $AB = BA$   
(d) If  $\det(A) = 2$  and  $\det(B) = 3$ , then  $\det(A + B) = 5$ .  
FALSE. In general,  $\det(A + B) \neq \det(A) + \det(B)$ .  
(e) Let  $A$  be  $n \times n$ . Then  $\det(A^T A) \geq 0$ .  
TRUE:  $\det(A^T A) = \det(A^T)\det(A) = (\det(A))^2$ .  
(f) If  $A^3 = 0$ , then  $\det(A) = 0$ .  
TRUE: If  $A^3 = 0$ , then  $(\det(A))^3 = 0$ , so  $\det(A) = 0$ .